THE MATHSCOPE

All the best from
Vietnamese Problem Solving Journals

`Updated November 2, 2005`

translated by Pham Van Thuan, Eckard Specht

Vol I, Problems in Mathematics Journal for the Youth

Mathscope is a free problem resource selected from problem solving journals in Vietnam. This freely accessible collection is our effort to introduce elementary mathematics problems to our foreign friends for either recreational or professional use. We would like to give you a new taste of Vietnamese mathematical culture. Whatever the purpose, we welcome suggestions and comments from you all. More communications can be addressed to *Pham Van Thuan, 4E2, 565 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam*, or email us at `pvthuan@vnu.edu.vn`.

It's now not too hard to find problems and solutions on the Internet due to the increasing numbers of websites devoted to mathematical problems solving. Anyway, we hope that this complete collection saves you considerable time searching the problems you really want. We intend to give an outline of solutions to the problems, but it would take time. Now enjoy these "cakes" from Vietnam first.

---

261. 1 (Ho Quang Vinh) Given a triangle ABC, its internal angle bisectors BE and CF, and let M be any point on the line segment EF. Denote by S_A, S_B, and S_C the areas of triangles MBC, MCA, and MAB, respectively. Prove that

$$\frac{\sqrt{S_B}+\sqrt{S_C}}{\sqrt{S_A}} \leq \sqrt{\frac{AC+AB}{BC}},$$

and determine when equality holds.

261. 2 (Editorial Board) Find the maximum value of the expression

$$A = 13\sqrt{x^2 - x^4} + 9\sqrt{x^2 + x^4} \qquad \text{for} \quad 0 \le x \le 1.$$

261. 3 (Editorial Board) The sequence (a_n), $n = 1, 2, 3, \ldots$, is defined by $a_1 > 0$, and $a_{n+1} = ca_n^2 + a_n$ for $n = 1, 2, 3, \ldots$, where c is a constant. Prove that

a) $a_n \ge \sqrt{c^{n-1} n^n a_1^{n+1}}$, and

b) $a_1 + a_2 + \cdots + a_n > n\left(na_1 - \dfrac{1}{c}\right)$ for $n \in \mathbb{N}$.

261. 4 (Editorial Board) Let X, Y, Z be the reflections of A, B, and C across the lines BC, CA, and AB, respectively. Prove that X, Y, and Z are collinear if and only if

$$\cos A \cos B \cos C = -\frac{3}{8}.$$

261. 5 (Vinh Competition) Prove that if $x, y, z > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ then the following inequality holds:

$$\left(1 - \frac{1}{1+x^2}\right)\left(1 - \frac{1}{1+y^2}\right)\left(1 - \frac{1}{1+z^2}\right) > \frac{1}{2}.$$

261. 6 (Do Van Duc) Given four real numbers x_1, x_2, x_3, x_4 such that $x_1 + x_2 + x_3 + x_4 = 0$ and $|x_1| + |x_2| + |x_3| + |x_4| = 1$, find the maximum value of $\prod\limits_{1 \le i < j \le 4} (x_i - x_j)$.

261. 7 (Doan Quang Manh) Given a rational number $x \ge 1$ such that there exists a sequence of integers (a_n), $n = 0, 1, 2, \ldots$, and a constant $c \ne 0$ such that $\lim\limits_{n \to \infty} (cx^n - a_n) = 0$. Prove that x is an integer.

262. 1 (Ngo Van Hiep) Let ABC an equilateral triangle of side length a. For each point M in the interior of the triangle, choose points D, E, F on the sides CA, AB, and BC, respectively, such that $DE = MA$, $EF = MB$, and $FD = MC$. Determine M such that $\triangle DEF$ has smallest possible area and calculate this area in terms of a.

262. 2 (Nguyen Xuan Hung) Given is an acute triangle with altitude AH. Let D be any point on the line segment AH not coinciding with the endpoints of this segment and the orthocenter of triangle ABC. Let ray BD intersect AC at M, ray CD meet AB at N. The line perpendicular to BM at M meets the line perpendicular to CN at N in the point S. Prove that $\triangle ABC$ is isosceles with base BC if and only if S is on line AH.

262. 3 (Nguyen Duy Lien) The sequence (a_n) is defined by

$$a_0 = 2, \quad a_{n+1} = 4a_n + \sqrt{15a_n^2 - 60} \quad \text{for } n \in \mathbb{N}.$$

Find the general term a_n. Prove that $\frac{1}{5}(a_{2n} + 8)$ can be expressed as the sum of squares of three consecutive integers for $n \geq 1$.

262. 4 (Tuan Anh) Let p be a prime, n and k positive integers with $k > 1$. Suppose that $b_i, i = 1, 2, \ldots, k$, are integers such that

i) $0 \leq b_i \leq k - 1 \quad$ for all i,

ii) p^{nk-1} is a divisor of $\left(\sum_{i=1}^{k} p^{nb_i}\right) - p^{n(k-1)} - p^{n(k-2)} - \cdots - p^n - 1.$

Prove that the sequence $(b_1, b_2, \ldots, b_k)$ is a permutation of the sequence $(0, 1, \ldots, k-1)$.

262. 5 (Doan The Phiet) Without use of any calculator, determine

$$\sin\frac{\pi}{14} + 6\sin^2\frac{\pi}{14} - 8\sin^4\frac{\pi}{14}.$$

264. 1 (Tran Duy Hinh) Prove that the sum of all squares of the divisors of a natural number n is less than $n^2\sqrt{n}$.

264. 2 (Hoang Ngoc Canh) Given two polynomials

$$f(x) = x^4 - (1 + \mathrm{e}^x) + \mathrm{e}^2, \qquad g(x) = x^4 - 1,$$

prove that for distinct positive numbers a, b satisfying $a^b = b^a$, we have $f(a)f(b) < 0$ and $g(a)g(b) > 0$.

264. 3 (Nguyen Phu Yen) Solve the equation

$$\frac{(x-1)^4}{(x^2-3)^2} + (x^2-3)^4 + \frac{1}{(x-1)^2} = 3x^2 - 2x - 5.$$

264. 4 (Nguyen Minh Phuong, Nguyen Xuan Hung) Let I be the incenter of triangle ABC. Rays AI, BI, and CI meet the circumcircle of triangle ABC again at X, Y, and Z, respectively. Prove that

a) $IX + IY + IZ \geq IA + IB + IC, \qquad$ b) $\dfrac{1}{IX} + \dfrac{1}{IY} + \dfrac{1}{IZ} \geq \dfrac{3}{R}.$

265. 1 (Vu Dinh Hoa) The lengths of the four sides of a convex quadrilateral are natural numbers such that the sum of any three of them is divisible by the fourth number. Prove that the quadrilateral has two equal sides.

265. 2 (Dam Van Nhi) Let AD, BE, and CF be the internal angle bisectors of triangle ABC. Prove that $p(DEF) \leq \frac{1}{2}p(ABC)$, where $p(XYZ)$ denotes the perimeter of triangle XYZ. When does equality hold?

266. 1 (Le Quang Nam) Given real numbers $x, y, z \geq -1$ satisfying $x^3 + y^3 + z^3 \geq x^2 + y^2 + z^2$, prove that $x^5 + y^5 + z^5 \geq x^2 + y^2 + z^2$.

266. 2 (Dang Nhon) Let $ABCD$ be a rhombus with $\angle A = 120°$. A ray Ax and AB make an angle of $15°$, and Ax meets BC and CD at M and N, respectively. Prove that

$$\frac{3}{AM^2} + \frac{3}{AN^2} = \frac{4}{AB^2}.$$

266. 3 (Ha Duy Hung) Given an isosceles triangle with $\angle A = 90°$. Let M be a variable point on line BC, (M distinct from B, C). Let H and K be the orthogonal projections of M onto lines AB and AC, respectively. Suppose that I is the intersection of lines CH and BK. Prove that the line MI has a fixed point.

266. 4 (Luu Xuan Tinh) Let x, y be real numbers in the interval $(0, 1)$ and $x + y = 1$, find the minimum of the expression $x^x + y^y$.

267. 1 (Do Thanh Han) Let x, y, z be real numbers such that

$$x^2 + z^2 = 1,$$
$$y^2 + 2y(x + z) = 6.$$

Prove that $y(z - x) \leq 4$, and determine when equality holds.

267. 2 (Le Quoc Han) In triangle ABC, medians AM and CN meet at G. Prove that the quadrilateral $BMGN$ has an incircle if and only if triangle ABC is isosceles at B.

267. 3 (Tran Nam Dung) In triangle ABC, denote by a, b, c the side lengths, and F the area. Prove that

$$F \leq \frac{1}{16}(3a^2 + 2b^2 + 2c^2),$$

and determine when equality holds. Can we find another set of the coefficients of a^2, b^2, and c^2 for which equality holds?

268. 1 (Do Kim Son) In a triangle, denote by a, b, c the side lengths, and let r, R be the inradius and circumradius, respectively. Prove that

$$a(b + c - a)^2 + b(c + a - b)^2 + c(a + b - c)^2 \leq 6\sqrt{3}R^2(2R - r).$$

268. 2 (Dang Hung Thang) The sequence (a_n), $n \in \mathbb{N}$, is defined by

$$a_0 = a, \quad a_1 = b, \quad a_{n+2} = da_{n+1} - a_n \qquad \text{for} \quad n = 0, 1, 2, \ldots,$$

where a, b are non-zero integers, d is a real number. Find all d such that a_n is an integer for $n = 0, 1, 2, \ldots$.

271. 1 (Doan The Phiet) Find necessary and sufficient conditions with respect to m such that the system of equations

$$\begin{aligned} &x^2 + y^2 + z^2 + xy - yz - zx = 1, \\ &y^2 + z^2 + yz = 2, \\ &z^2 + x^2 + zx = m \end{aligned}$$

has a solution.

272. 1 (Nguyen Xuan Hung) Given are three externally tangent circles $(O_1), (O_2)$, and (O_3). Let A, B, C be respectively the points of tangency of (O_1) and (O_3), (O_2) and (O_3), (O_1) and (O_2). The common tangent of (O_1) and (O_2) meets C and (O_3) at M and N. Let D be the midpoint of MN. Prove that C is the center of one of the excircles of triangle ABD.

272. 2 (Trinh Bang Giang) Let $ABCD$ be a convex quadrilateral such that $AB + CD = BC + DA$. Find the locus of points M interior to quadrilateral $ABCD$ such that the sum of the distances from M to AB and CD is equal to the sum of the distances from M to BC and DA.

272. 3 (Ho Quang Vinh) Let M and m be the greatest and smallest numbers in the set of positive numbers $a_1, a_2, \ldots, a_n$, $n \geq 2$. Prove that

$$\Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{i=1}^{n} \frac{1}{a_i}\Big) \leq n^2 + \frac{n(n-1)}{2}\Big(\sqrt{\frac{M}{m}} - \sqrt{\frac{m}{M}}\Big)^2.$$

272. 4 (Nguyen Huu Du) Find all primes p such that

$$f(p) = (2+3) - (2^2+3^2) + (2^3+3^3) - \cdots - (2^{p-1}+3^{p-1}) + (2^p+3^p)$$

is divisible by 5.

274. 1 (Dao Manh Thang) Let p be the semiperimeter and R the circumradius of triangle ABC. Furthermore, let D, E, F be the excenters. Prove that

$$DE^2 + EF^2 + FD^2 \geq 8\sqrt{3}pR,$$

and determine the equality case.

274. 2 (Doan The Phiet) Detemine the positive root of the equation

$$x\ln\Big(1+\frac{1}{x}\Big)^{1+\frac{1}{x}} - x^3\ln\Big(1+\frac{1}{x^2}\Big)^{1+\frac{1}{x^2}} = 1-x.$$

274. 3 (N.Khanh Nguyen) Let $ABCD$ be a cyclic quadrilateral. Points M, N, P, and Q are chosen on the sides AB, BC, CD, and DA, respectively, such that $MA/MB = PD/PC = AD/BC$ and $QA/QD = NB/NC = AB/CD$. Prove that MP is perpendicular to NQ.

274. 4 (Nguyen Hao Lieu) Prove the inequality for $x \in \mathbb{R}$:

$$\frac{1+2x\arctan x}{2+\ln(1+x^2)^2} \geq \frac{1+\mathrm{e}^{\frac{x}{2}}}{3+\mathrm{e}^x}.$$

275. 1 (Tran Hong Son) Let x, y, z be real numbers in the interval $[-2, 2]$, prove the inequality

$$2(x^6+y^6+z^6) - (x^4y^2+y^4z^2+z^4x^2) \leq 192.$$

276. 1 (Vu Duc Canh) Find the maximum value of the expression

$$f = \frac{a^3+b^3+c^3}{abc},$$

where a, b, c are real numbers lying in the interval $[1, 2]$.

276. 2 (Ho Quang Vinh) Given a triangle ABC with sides $BC = a$, $CA = b$, and $AB = c$. Let R and r be the circumradius and inradius of the triangle, respectively. Prove that

$$\frac{a^3+b^3+c^3}{abc} \geq 4 - \frac{2r}{R}.$$

276. 3 (Pham Hoang Ha) Given a triangle ABC, let P be a point on the side BC, let H, K be the orthogonal projections of P onto AB, AC respectively. Points M, N are chosen on AB, AC such that $PM \parallel AC$ and $PN \parallel AB$. Compare the areas of triangles PHK and PMN.

276. 4 (Do Thanh Han) How many 6-digit natural numbers exist with the distinct digits and two arbitrary consecutive digits can not be simultaneously odd numbers?

277. 1 (Nguyen Hoi) The incircle with center O of a triangle touches the sides AB, AC, and BC respectively at D, E, and F. The escribed circle of triangle ABC in the angle A has center Q and touches the side BC and the rays AB, AC respectively at K, H, and I. The line DE meets the rays BO and CO respectively at M and N. The line HI meets the rays BQ and CQ at R and S, respectively. Prove that

a) $\triangle FMN = \triangle KRS$, b) $\dfrac{IS}{AB} = \dfrac{SR}{BC} = \dfrac{RH}{CA}$.

277. 2 (Nguyen Duc Huy) Find all rational numbers p, q, r such that

$$p \cos \frac{\pi}{7} + q \cos \frac{2\pi}{7} + r \cos \frac{3\pi}{7} = 1.$$

277. 3 (Nguyen Xuan Hung) Let $ABCD$ be a bicentric quadrilateral inscribed in a circle with center I and circumcribed about a circle with center O. A line through I, parallel to a side of $ABCD$, intersects its two opposite sides at M and N. Prove that the length of MN does not depend on the choice of side to which the line is parallel.

277. 4 (Dinh Thanh Trung) Let $x \in (0, \pi)$ be real number and suppose that $\frac{x}{\pi}$ is not rational. Define

$$S_1 = \sin x, \;\; S_2 = \sin x + \sin 2x, \;\; \ldots, \;\; S_n = \sin x + \sin 2x + \cdots + \sin nx.$$

Let t_n be the number of negative terms in the sequence $S_1, S_2, \ldots, S_n$. Prove that $\lim\limits_{n\to\infty} \frac{t_n}{n} = \frac{x}{2\pi}$.

279. 1 (Nguyen Huu Bang) Find all natural numbers $a > 1$, such that if p is a prime divisor of a then the number of all divisors of a which are relatively prime to p, is equal to the number of the divisors of a that are not relatively prime to p.

279. 2 (Le Duy Ninh) Prove that for all real numbers a, b, x, y satisfying $x + y = a + b$ and $x^4 + y^4 = a^4 + b^4$ then $x^n + y^n = a^n + b^n$ for all $n \in \mathbb{N}$.

279. 3 (Nguyen Huu Phuoc) Given an equilateral triangle ABC, find the locus of points M interior to ABC such that if the orthogonal projections of M onto BC, CA and AB are D, E, and F, respectively, then AD, BE, and CF are concurrent.

279. 4 (Nguyen Minh Ha) Let M be a point in the interior of triangle ABC and let X, Y, Z be the reflections of M across the sides BC, CA, and AB, respectively. Prove that triangles ABC and XYZ have the same centroid.

279. 5 (Vu Duc Son) Find all positive integers n such that $n < t_n$, where t_n is the number of positive divisors of n^2.

279. 6 (Tran Nam Dung) Find the maximum value of the expression

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2},$$

where x, y, z are real numbers satisfying the condition $x + y + z = 1$.

279. 7 (Hoang Hoa Trai) Given are three concentric circles with center O, and radii $r_1 = 1$, $r_2 = \sqrt{2}$, and $r_3 = \sqrt{5}$. Let A, B, C be three non-collinear points lying respectively on these circles and let F be the area of triangle ABC. Prove that $F \leq 3$, and determine the side lengths of triangle ABC.

281. 1 (Nguyen Xuan Hung) Let P be a point exterior to a circle with center O. From P construct two tangents touching the circle at A and B. Let Q be a point, distinct from P, on the circle. The tangent at Q of the circle intersects AB and AC at E and F, respectively. Let BC intersect OE and OF at X and Y, respectively. Prove that XY/EF is a constant when P varies on the circle.

281. 2 (Ho Quang Vinh) In a triangle ABC, let $BC = a$, $CA = b$, $AB = c$ be the sides, r, r_a, r_b, and r_c be the inradius and exradii. Prove that

$$\frac{abc}{r} \geq \frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c}.$$

283. 1 (Tran Hong Son) Simplify the expression

$$\sqrt{x(4-y)(4-z)} + \sqrt{y(4-z)(4-x)} + \sqrt{z(4-x)(4-y)} - \sqrt{xyz},$$

where x, y, z are positive numbers such that $x + y + z + \sqrt{xyz} = 4$.

283. 2 (Nguyen Phuoc) Let $ABCD$ be a convex quadrilateral, M be the midpoint of AB. Point P is chosen on the segment AC such that lines MP and BC intersect at T. Suppose that Q is on the segment BD such that $BQ/QD = AP/PC$. Prove that the line TQ has a fixed point when P moves on the segment AC.

284. 1 (Nguyen Huu Bang) Given an integer $n > 0$ and a prime $p > n + 1$, prove or disprove that the following equation has integer solutions:

$$1 + \frac{x}{n+1} + \frac{x^2}{2n+1} + \cdots + \frac{x^p}{pn+1} = 0.$$

284. 2 (Le Quang Nam) Let x, y be real numbers such that

$$(x + \sqrt{1+y^2})(y + \sqrt{1+x^2}) = 1,$$

prove that

$$(x + \sqrt{1+x^2})(y + \sqrt{1+y^2}) = 1.$$

284. 3 (Nguyen Xuan Hung) The internal angle bisectors AD, BE, and CF of a triangle ABC meet at point Q. Prove that if the inradii of triangles AQF, BQD, and CQE are equal then triangle ABC is equilateral.

284. 4 (Tran Nam Dung) Disprove that there exists a polynomial $p(x)$ of degree greater than 1 such that if $p(x)$ is an integer then $p(x+1)$ is also an integer for $x \in \mathbb{R}$.

285. 1 (Nguyen Duy Lien) Given an odd natural number p and integers a, b, c, d, e such that $a+b+c+d+e$ and $a^2+b^2+c^2+d^2+e^2$ are all divisible by p. Prove that $a^5+b^5+c^5+d^5+e^5-5abcde$ is also divisible by p.

285. 2 (Vu Duc Canh) Prove that if $x, y \in \mathbb{R}^*$ then

$$\frac{2x^2+3y^2}{2x^3+3y^3} + \frac{2y^2+3x^2}{2y^3+3x^3} \leq \frac{4}{x+y}.$$

285. 3 (Nguyen Huu Phuoc) Let P be a point in the interior of triangle ABC. Rays AP, BP, and CP intersect the sides BC, CA, and AB at D, E, and F, respectively. Let K be the point of intersection of DE and CM, H be the point of intersection of DF and BM. Prove that AD, BK and CH are concurrent.

285. 4 (Tran Tuan Anh) Let a, b, c be non-negative real numbers, determine all real numbers x such that the following inequality holds:

$$\begin{aligned}[][a^2+b^2+(x-1)c^2][a^2+c^2+(x-1)b^2][b^2+c^2+(x-1)a^2] \\ \leq (a^2+xbc)(b^2+xac)(c^2+xab).\end{aligned}$$

285. 5 (Truong Cao Dung) Let O and I be the circumcenter and incenter of a triangle ABC. Rays AI, BI, and CI meet the circumcircle at D, E, and F, respectively. Let R_a, R_b, and R_c be the radii of the escribed circles of $\triangle ABC$, and let R_d, R_e, and R_f be the radii of the escribed circles of triangle DEF. Prove that

$$R_a + R_b + R_c \leq R_d + R_e + R_f.$$

285. 6 (Do Quang Duong) Determine all integers k such that the sequence defined by $a_1 = 1$, $a_{n+1} = 5a_n + \sqrt{ka_n^2 - 8}$ for $n = 1, 2, 3, \ldots$ includes only integers.

286. 1 (Tran Hong Son) Solve the equation

$$18x^2 - 18x\sqrt{x} - 17x - 8\sqrt{x} - 2 = 0.$$

286. 2 (Pham Hung) Let $ABCD$ be a square. Points E, F are chosen on CB and CD, respectively, such that $BE/BC = k$, and $DF/DC = (1-k)/(1+k)$, where k is a given number, $0 < k < 1$. Segment BD meets AE and AF at H and G, respectively. The line through A, perpendicular to EF, intersects BD at P. Prove that $PG/PH = DG/BH$.

286. 3 (Vu Dinh Hoa) In a convex hexagon, the segment joining two of its vertices, dividing the hexagon into two quadrilaterals is called a *principal* diagonal. Prove that in every convex hexagon, in which the length of each side is equal to 1, there exists a principal diagonal with length not greater than 2 and there exists a principal diagonal with length greater than $\sqrt{3}$.

286. 4 (Do Ba Chu) Prove that in any acute or right triangle ABC the following inequality holds:

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \geq \frac{10\sqrt{3}}{9}.$$

286. 5 (Tran Tuan Diep) In triangle ABC, no angle exceeding $\frac{\pi}{2}$, and each angle is greater than $\frac{\pi}{4}$. Prove that

$$\cot A + \cot B + \cot C + 3\cot A\cot B\cot C \leq 4(2-\sqrt{2}).$$

287. 1 (Tran Nam Dung) Suppose that a, b are positive integers such that $2a-1$, $2b-1$ and $a+b$ are all primes. Prove that $a^b + b^a$ and $a^a + b^b$ are not divisible by $a+b$.

287. 2 (Pham Dinh Truong) Let $ABCD$ be a square in which the two diagonals intersect at E. A line through A meets BC at M and intersects CD at N. Let K be the intersection point of EM and BN. Prove that $CK \perp BN$.

287. 3 (Nguyen Xuan Hung) Let ABC be a right isosceles triangle, $\angle A = 90^\circ$, I be the incenter of the triangle, M be the midpoint of BC. Let MI intersect AB at N and E be the midpoint of IN. Furthermore, F is chosen on side BC such that $FC = 3FB$. Suppose that the line EF intersects AB and AC at D and K, respectively. Prove that $\triangle ADK$ is isosceles.

287. 4 (Hoang Hoa Trai) Given a positive integer n, and w is the sum of n first integers. Prove that the equation

$$x^3 + y^3 + z^3 + t^3 = 2w^3 - 1$$

has infinitely many integer solutions.

288. 1 (Vu Duc Canh) Find necessary and sufficient conditions for a, b, c for which the following equation has no solutions:

$$a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c = x.$$

288. 2 (Pham Ngoc Quang) Let $ABCD$ be a cyclic quadrilateral, P be a variable point on the arc BC not containing A, and F be the foot of the perpendicular from C onto AB. Suppose that $\triangle MEF$ is equilateral, calculate IK/R, where I is the incenter of triangle ABC and K the intersection (distinct from A) of ray AI and the circumcircle of radius R of triangle ABC.

288. 3 (Nguyen Van Thong) Given a prime $p > 2$ such that $p - 2$ is divisible by 3. Prove that the set of integers defined by $y^2 - x^3 - 1$, where x, y are non-negative integers smaller than p, has at most $p - 1$ elements divisible by p.

289. 1 (Thai Nhat Phuong) Let ABC be a right isosceles triangle with $A = 90°$. Let M be the midpoint of BC, G be a point on side AB such that $GB = 2GA$. Let GM intersect CA at D. The line through M, perpendicular to CG at E, intersects AC at K. Finally, let P be the point of intersection of DE and GK. Prove that $DE = BC$ and $PG = PE$.

289. 2 (Ho Quang Vinh) Given a convex quadrilateral $ABCD$, let M and N be the midpoints of AD and BC, respectively, P be the point of intersection of AN and BM, and Q the intersection point of DN and CM. Prove that

$$\frac{PA}{PN} + \frac{PB}{PM} + \frac{QC}{QM} + \frac{QD}{QN} \geq 4,$$

and determine when equality holds.

290. 1 (Nguyen Song Minh) Given $x, y, z, t \in \mathbb{R}$ and real polynomial

$$F(x, y, z, t) = 9(x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2) + 6xz(y^2 + t^2) - 4xyzt.$$

a) Prove that the polynomial can be factored into the product of two quadratic polynomials.

b) Find the minimum value of the polynomial F if $xy + zt = 1$.

290. 2 (Pham Hoang Ha) Let M be a point on the internal angle bisector AD of triangle ABC, M distinct from A, D. Ray AM intersects side AC at E, ray CM meets side AB at F. Prove that if

$$\frac{1}{AB^2} + \frac{1}{AE^2} = \frac{1}{AC^2} + \frac{1}{AF^2}$$

then $\triangle ABC$ is isosceles.

290. 3 (Do Anh) Consider a triangle ABC and its incircle. The internal angle bisector AD and median AM intersect the incircle again at P and Q, respectively. Compare the lengths of DP and MQ.

290. 4 (Nguyen Duy Lien) Find all pairs of integers (a, b) such that $a + b^2$ divides $a^2b - 1$.

290. 5 (Dinh Thanh Trung) Determine all real functions $f(x)$, $g(x)$ such that $f(x) - f(y) = \cos(x + y) \cdot g(x - y)$ for all $x, y \in \mathbb{R}$.

290. 6 (Nguyen Minh Duc) Find all real numbers a such that the system of equations has real solutions in x, y, z:

$$\begin{aligned}
&\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} = a - 1, \\
&\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} = a + 1.
\end{aligned}$$

290. 7 (Doan Kim Sang) Given a positive integer n, find the number of positive integers, not exceeding $n(n + 1)(n + 2)$, which are divisible by n, $n + 1$, and $n + 2$.

291. 1 (Bui Minh Duy) Given three distinct numbers a, b, c such that

$$\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0,$$

prove that any two of the numbers have different signs.

291. 2 (Do Thanh Han) Given three real numbers x, y, z that satisfy the conditions $0 < x < y \leq z \leq 1$ and $3x + 2y + z \leq 4$. Find the maximum value of the expression $3x^3 + 2y^2 + z^2$.

291. 3 (Vi Quoc Dung) Given a circle of center O and two points A, B on the circle. A variable circle through A, B has center Q. Let P be the reflection of Q across the line AB. Line AP intersects the circle O again at E, while line BE, E distinct from B, intersects the circle Q again at F. Prove that F lies on a fixed line when circle Q varies.

291. 4 (Vu Duc Son) Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(f(x) + y) = x + f(y) \quad \text{for } x, y \in \mathbb{Q}.$$

291. 5 (Nguyen Van Thong) Find the maximum value of the expression

$$x^2(y - z) + y^2(z - y) + z^2(1 - z),$$

where x, y, z are real numbers such that $0 \leq x \leq y \leq z \leq 1$.

291. 6 (Vu Thanh Long) Given an acute-angled triangle ABC with side lengths a, b, c. Let R, r denote its circumradius and inradius, respectively, and F its area. Prove the inequality

$$ab + bc + ca \geq 2R^2 + 2Rr + \frac{8}{\sqrt{3}}F.$$

292. 1 (Thai Nhat Phuong, Tran Ha) Let x, y, z be positive numbers such that $xyz = 1$, prove the inequality

$$\frac{x^2}{x + y + y^3 z} + \frac{y^2}{y + z + z^3 x} + \frac{z^2}{z + x + x^3 y} \leq 1.$$

292. 2 (Pham Ngoc Boi) Let p be an odd prime, let $a_1, a_2, \ldots, a_{p-1}$ be $p - 1$ integers that are not divisible by p. Prove that among the sums $T = k_1 a_1 + k_2 a_2 + \cdots + k_{p-1} a_{p-1}$, where $k_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, p-1$, there exists at least a sum T divisible by p.

292. 3 (Ha Vu Anh) Given are two circles Γ_1 and Γ_2 intersecting at two distinct points A, B and a variable point P on Γ_1, P distinct from A and B. The lines PA, PB intersect Γ_2 at D and E, respectively. Let M be the midpoint of DE. Prove that the line MP has a fixed point.

294. 1 (Phung Trong Thuc) Triangle ABC is inscribed in a circle of center O. Let M be a point on side AC, M distinct from A, C, the line BM meets the circle again at N. Let Q be the intersection of a line through A perpendicular to AB and a line through N perpendicular to NC. Prove that the line QM has a fixed point when M varies on AC.

294. 2 (Tran Xuan Bang) Let A, B be the intersections of circle O of radius R and circle O' of radius R'. A line touches circle O and O' at T and T', respectively. Prove that B is the centroid of triangle ATT' if and only if

$$OO' = \frac{\sqrt{3}}{2}(R + R').$$

294. 3 (Vu Tri Duc) If a, b, c are positive real numbers such that $ab + bc + ca = 1$, find the minimum value of the expression $w(a^2 + b^2) + c^2$, where w is a positive real number.

294. 4 (Le Quang Nam) Let p be a prime greater than 3, prove that $\binom{p-1}{2001p^2-1} - 1$ is divisible by p^4.

294. 5 (Truong Ngoc Dac) Let x, y, z be positive real numbers such that $x = \max\{x, y, z\}$, find the minimum value of

$$\frac{x}{y} + \sqrt{1 + \frac{y}{z}} + \sqrt[3]{1 + \frac{z}{x}}.$$

294. 6 (Pham Hoang Ha) The sequence (a_n), $n = 1, 2, 3, \ldots$, is defined by $a_n = \frac{1}{n^2(n+2)\sqrt{n+1}}$ for $n = 1, 2, 3, \ldots$.

Prove that

$$a_1 + a_2 + \cdots + a_n < \frac{1}{2\sqrt{2}} \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

294. 7 (Vu Huy Hoang) Given are a circle O of radius R, and an odd natural number n. Find the positions of n points $A_1, A_2, \ldots, A_n$ on the circle such that the sum $A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n + A_nA_1$ is a minimum.

295. 1 (Tran Tuyet Thanh) Solve the equation

$$x^2 - x - 1000\sqrt{1 + 8000x} = 1000.$$

295. 2 (Pham Dinh Truong) Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon with parallel opposite sides. Let B_1, B_2, and B_3 be the points of intersection of pairs of diagonals A_1A_4 and A_2A_5, A_2A_5 and A_3A_6, A_3A_6 and A_1A_4, respectively. Let C_1, C_2, C_3 be respectively the midpoints of the segments A_3A_6, A_1A_4, A_2A_5. Prove that B_1C_1, B_2C_2, B_3C_3 are concurrent.

295. 3 (Bui The Hung) Let A, B be respectively the greatest and smallest numbers from the set of n positive numbers $x_1, x_2, \ldots, x_n$, $n \geq 2$. Prove that

$$A < \frac{(x_1 + x_2 + \cdots + x_n)^2}{x_1 + 2x_2 + \cdots + nx_n} < 2B.$$

295. 4 (Tran Tuan Anh) Prove that if $x, y, z > 0$ then

a) $(x + y + z)^3(y + z - x)(z + x - y)(x + y - z) \leq 27x^3y^3z^3$,

b) $(x^2 + y^2 + z^2)(y + z - x)(z + x - y)(x + y - z) \leq xyz(yz + zx + xy)$,

c) $(x + y + z)\left[2(yz + zx + xy) - (x^2 + y^2 + z^2)\right] \leq 9xyz$.

295. 5 (Vu Thi Hue Phuong) Find all functions $f : \mathbb{D} \to \mathbb{D}$, where $\mathbb{D} = [1, +\infty)$ such that

$$f(xf(y)) = yf(x) \quad \text{for } x, y \in \mathbb{D}.$$

295. 6 (Nguyen Viet Long) Given an even natural number n, find all polynomials $p_n(x)$ of degree n such that

i) all the coefficients of $p_n(x)$ are elements from the set $\{0, -1, 1\}$ and $p_n(0) \neq 0$;

ii) there exists a polynomial $q(x)$ with coefficients from the set $\{0, -1, 1\}$ such that $p_n(x) \equiv (x^2 - 1)q(x)$.

296. 1 (Thoi Ngoc Anh) Prove that

$$\frac{1}{6} < \frac{3 - \underbrace{\sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}}_{n \text{ times}}}{3 - \underbrace{\sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}}_{(n-1) \text{ times}}} < \frac{5}{27},$$

where there are n radical signs in the expression of the numerator and $n - 1$ ones in the expression of the denominator.

296. 2 (Vi Quoc Dung) Let ABC be a triangle and M the midpoint of BC. The external angle bisector of A meets BC at D. The circumcircle of triangle ADM intersects line AB and line AC at E and F, respectively. If N is the midpoint of EF, prove that $MN \parallel AD$.

296. 3 (Nguyen Van Hien) Let $k, n \in \mathbb{N}$ such that $k < n$. Prove that

$$\frac{(n+1)^{n+1}}{(k+1)^{k+1}(n-k+1)^{n-k+1}} < \frac{n!}{k!\,(n-k)!} < \frac{n^n}{k^k(n-k+1)^{n-k}}.$$

297. 1 (Nguyen Huu Phuoc) Given a circle with center O and diameter EF. Points N, P are chosen on line EF such that $ON = OP$. From a point M interior to the circle, not lying on EF, draw MN intersecting the circle at A and C, draw MP meeting the circle at B and D such that B and O are on different sides of AC. Let K be the point of intersection of OB and AC, Q the point of intersection of EF and CD. Prove that lines KQ, BD, AO are concurrent.

297. 2 (Tran Nam Dung) Let a and b two relatively prime numbers. Prove that there exist exactly $\frac{1}{2}(ab - a - b + 1)$ natural numbers that can not be written in the form $ax + by$, where x and y are non-negative integers.

297. 3 (Le Quoc Han) The circle with center I and radius r touches the sides $BC = a$, $CA = b$, and $AB = c$ of triangle ABC at M, N, and P, respectively. Let F be the area of triangle ABC and h_a, h_b, h_c be the lengths of the altitudes of $\triangle ABC$. Prove that

a) $4F^2 = ab \cdot MN^2 + bc \cdot NP^2 + ca \cdot PM^2$;

b) $\dfrac{MN^2}{h_a h_b} + \dfrac{NP^2}{h_b h_c} + \dfrac{PM^2}{h_c h_a} = 1.$

298. 1 (Pham Hoang Ha) Let P be the midpoint of side BC of triangle ABC and let BE, CF be two altitudes of the triangle. The line through A, perpendicular to PF, meets CF at M; the line through A, perpendicular to PE, intersects BE at N. Let K and G be respectively the midpoints of BM and CN. Finally, let H be the intersection of KF and GE. Prove that AH is perpendicular to EF.

298. 2 (Pham Dinh Truong) Let $ABCD$ be a square. Points E and F are chosen on sides AB and CD, respectively, such that $AE = CF$. Let AD intersect CE and BF at M and N, respectively. Suppose that P is the intersection of BM and CN, find the locus of P when E and F move on the side AB and CD, respectively.

298. 3 (Nguyen Minh Ha) Let $ABCD$ be a convex quadrilateral, let AB intersect CD at E; AD intersects BC at F. Prove that the midpoints of line segments AB, CD, and EF are collinear.

298. 4 (Nguyen Minh Ha) Given a cylic quadrilateral $ABCD$, M is any point in the plane. Let X, Y, Z, T, U, V be the orthogonal projections of M on the lines AB, CD, AC, DB, AD, and BC. Let E, F, G be the midpoints of XY, ZT, and UV. Prove that E, F, and G are collinear.

300. 1 (Vu Tri Duc) Find the maximum and minimum values of the expression $x\sqrt{1+y} + y\sqrt{1+x}$, where x, y are non-negative real numbers such that $x + y = 1$.

300. 2 (Nguyen Xuan Hung) Let P be a point in the interior of triangle ABC. The incircle of triangle ABC is tangent to sides BC, CA and AB at D, E, and F, respectively. The incircle of triangle PBC touches the sides BC, CP, and PB at K, M, and N, respectively. Suppose that Q is the point of intersection of lines EM and FN. Prove that A, P, Q are collinear if and only if K coincides with D.

300. 3 (Huynh Tan Chau) Determine all pairs of integers (m, n) such that

$$\frac{n}{m} = \frac{(m^2 - n^2)^{n/m} - 1}{(m^2 - n^2)^{n/m} + 1}.$$

300. 4 (Vo Giang Giai, Manh Tu) Prove that if $a, b, c, d, e \geq 0$ then

$$\frac{a+b+c+d+e}{5} \geq \sqrt[5]{abcde} + \frac{q}{20},$$

where $q = (\sqrt{a}-\sqrt{b})^2 + (\sqrt{b}-\sqrt{c})^2 + (\sqrt{c}-\sqrt{d})^2 + (\sqrt{d}-\sqrt{e})^2$.

306. 1 (Phan Thi Mui) Prove that if $x, y, z > 0$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$ then

$$(x+y-z-1)(y+z-x-1)(z+x-y-1) \leq 8.$$

306. 2 (Tran Tuan Anh) Given an integer $m \geq 4$, find the maximum and minimum values of the expression $ab^{m-1} + a^{m-1}b$, where a, b are real numbers such that $a + b = 1$ and $0 \leq a, b \leq \frac{m-2}{m}$.

308. 1 (Le Thi Anh Thu) Find all integer solutions of the equation

$$4(a-x)(x-b) + b - a = y^2,$$

where a, b are given integers, $a > b$.

308. 2 (Phan The Hai) Given a convex quadrilateral $ABCD$, E is the point of intersection of AB and CD, and F is the intersection of AD and BC. The diagonals AC and BD meet at O. Suppose that M, N, P, Q are the midpoints of AB, BC, CD, and DA. Let H be the intersection of OF and MP, and K the intersection of OE and NQ. Prove that $HK \parallel EF$.

309. 1 (Vu Hoang Hiep) Given a positive integer n, find the smallest possible $t = t(n)$ such that for all real numbers $x_1, x_2, \ldots, x_n$ we have

$$\sum_{k=1}^{n}(x_1 + x_2 + \cdots + x_k)^2 \leq t(x_1^2 + x_2^2 + \cdots + x_n^2).$$

309. 2 (Le Xuan Son) Given a triangle ABC, prove that

$$\sin A \cos B + \sin B \cos C + \sin C \cos A \leq \frac{3\sqrt{3}}{4}.$$

311. 1 (Nguyen Xuan Hung) The chord PQ of the circumcircle of a triangle ABC meets its incircle at M and N. Prove that $PQ \geq 2MN$.

311. 2 (Dam Van Nhi) Given a convex quadrilateral $ABCD$ with perpendicular diagonals AC and BD, let BC intersect AD at I and let AB meet CD at J. Prove that $BDIJ$ is cyclic if and only if $AB \cdot CD = AD \cdot BC$.

318. 1 (Dau Thi Hoang Oanh) Prove that if $2n$ is a sum of two distinct perfect square numbers (greater than 1) then $n^2 + 2n$ is the sum of four perfect square numbers (greater than 1).

318. 2 (Nguyen De) Solve the system of equations

$$x^2(y+z)^2 = (3x^2 + x + 1)y^2z^2,$$
$$y^2(z+x)^2 = (4y^2 + y + 1)z^2x^2,$$
$$z^2(x+y)^2 = (5z^2 + z + 1)x^2y^2.$$

318. 3 (Tran Viet Hung) A quadrilateral $ABCD$ is insribed in a circle such that the circle of diameter CD intersects the line segments AC, AD, BC, BD respectively at A_1, A_2, B_1, B_2, and the circle of diameter AB meets the line segments CA, CB, DA, DB respectively at C_1, C_2, D_1, D_2. Prove that there exists a circle that is tangent to the four lines A_1A_2, B_1B_2, C_1C_2 and D_1D_2.

319. 1 (Duong Chau Dinh) Prove the inequality

$$x^2y + y^2z + z^2x \le x^3 + y^3 + z^3 \le 1 + \frac{1}{2}(x^4 + y^4 + z^4),$$

where x, y, z are real non-negative numbers such that $x + y + z = 2$.

319. 2 (To Minh Hoang) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$2(f(m^2 + n^2))^3 = f^2(m)f(n) + f^2(n)f(m)$$

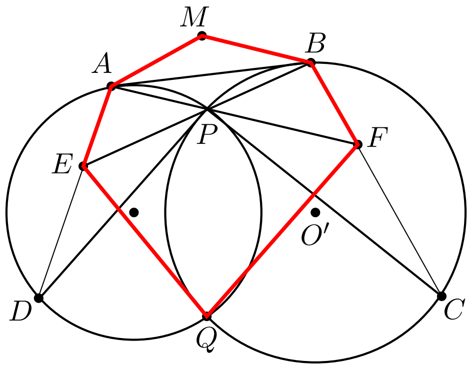
for distinct m and n.

319. 3 (Tran Viet Anh) Suppose that AD, BE and CF are the altitudes of an acute triangle ABC. Let M, N, and P be the intersection points of AD and EF, BE and FD, CF and DE respectively. Denote the area of triangle XYZ by $F[XYZ]$. Prove that

$$\frac{1}{F[ABC]} \le \frac{F[MNP]}{F^2[DEF]} \le \frac{1}{8\cos A \cos B \cos C \cdot F[ABC]}.$$

320. 1 (Nguyen Quang Long) Find the maximum value of the function $f = \sqrt{4x - x^3} + \sqrt{x + x^3}$ for $0 \le x \le 2$.

320. 2 (Vu Dinh Hoa) Two circles of centers O and O' intersect at P and Q (see Figure). The common tangent, adjacent to P, of the two circles touches O at A and O' at B. The tangent of circle O at P intersects O' at C; and the tangent of O' at P meets the circle O at D. Let M be the reflection of P across the midpoint of AB. The line AP intersects BC at E and the line BP meets AD at F. Prove that the hexagon $AMBEQF$ is cyclic.



320. 3 (Ho Quang Vinh) Let R and r be the circumradius and inradius of triangle ABC; the incircle touches the sides of the triangle at three points which form a triangle of perimeter p. Suppose that q is the perimeter of triangle ABC. Prove that $r/R \leq p/q \leq \frac{1}{2}$.

321. 1 (Le Thanh Hai) Prove that for all positive numbers a, b, c, d

a) $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} \geq \dfrac{a+b+c}{\sqrt[3]{abc}}$;

b) $\dfrac{a^2}{b^2} + \dfrac{b^2}{c^2} + \dfrac{c^2}{d^2} + \dfrac{d^2}{a^2} \geq \dfrac{a+b+c+d}{\sqrt[4]{abcd}}$.

321. 2 (Pham Hoang Ha) Find necessary and sufficient conditions for which the system of equations

$$x^2 = (2+m)y^3 - 3y^2 + my,$$
$$y^2 = (2+m)z^3 - 3z^2 + mz,$$
$$z^2 = (2+m)x^3 - 3x^2 + mx$$

has a unique solution.

321. 3 (Tran Viet Anh) Let m, n, p be three positive integers such that $n+1$ is divisible by m. Find a formula for the set of numbers $(x_1, x_2, \ldots, x_p)$ of p positive primes such that the sum $x_1 + x_2 + \cdots + x_p$ is divisible by m, with each number of the set not exceeding n.

322. 1 (Nguyen Nhu Hien) Given a triangle ABC with incenter I. The lines AI and DI intersect the circumcircle of triangle ABC again at H and K, respectively. Draw IJ perpendicular to BC at J. Prove that H, K and J are collinear.

322. 2 (Tran Tuan Anh) Prove the inequality

$$\frac{1}{2}\Big(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{x_i}\Big) \geq n - 1 + \frac{n}{\sum\limits_{i=1}^{n} x_i},$$

where x_i $(i = 1, 2, \ldots, n)$ are positive real numbers such that $\sum\limits_{i=1}^{n} x_i^2 = n$, with n as an integer, $n > 1$.

323. 1 (Nguyen Duc Thuan) Suppose that $ABCD$ is a convex quadrilateral. Points E, F are chosen on the lines BC and AD, respectively, such that $AE \parallel CD$ and $CF \parallel AB$. Prove that A, B, C, D are concyclic if and only if $AECF$ has an incircle.

323. 2 (Nguyen The Phiet) Prove that for an acute triangle ABC,

$$\cos A + \cos B + \cos C + \frac{1}{3}(\cos 3B + \cos 3C) \geq \frac{5}{6}.$$

324. 1 (Tran Nam Dung) Find the greatest possible real number c such that we can always choose a real number x which satisfies the inequality $\sin(mx) + \sin(nx) \geq c$ for each pair of positive integers m and n.

325. 1 (Nguyen Dang Phat) Given a convex hexagon inscribed in a circle such that the opposite sides are parallel. Prove that the sums of the lengths of each pair of opposite sides are equal if and only if the distances of the opposite sides are the same.

325. 2 (Dinh Van Kham) Given a natural number n and a prime p, how many sets of p natural numbers $\{a_0, a_1, \ldots, a_{p-1}\}$ are there such that

a) $1 \leq a_i \leq n$ for each $i = 0, 1, \ldots, p - 1$,

b) $[a_0, a_1, \ldots, a_{p-1}] = p \min\{a_0, a_1, \ldots, a_{p-1}\}$,

where $[a_0, a_1, \ldots, a_{p-1}]$ denotes the least common multiple of the numbers $a_0, a_1, \ldots, a_{p-1}$?

327. 1 (Hoang Trong Hao) Let $ABCD$ be a bicentric quadrilateral (i.e., it has a circumcircle of radius R and an incircle of radius r). Prove that $R \geq r\sqrt{2}$.

327. 2 (Vu Dinh The) Two sequences (x_n) and (y_n) are defined by

$$x_{n+1} = -2x_n^2 - 2x_n y_n + 8y_n^2, \quad x_1 = -1,$$
$$y_{n+1} = 2x_n^2 + 3x_n y_n - 2y_n^2, \quad y_1 = 1$$

for $n = 1, 2, 3, \ldots$. Find all primes p such that $x_p + y_p$ is not divisible by p.

328. 1 (Bui Van Chi) Find all integer solutions (n, m) of the equation

$$(n+1)(2n+1) = 10m^2.$$

328. 2 (Nguyen Thi Minh) Determine all positive integers n such that the polynomial of $n+1$ terms

$$p(x) = x^{4n} + x^{4(n-1)} + \cdots + x^8 + x^4 + 1$$

is divisible by the polynomial of $n+1$ terms

$$q(x) = x^{2x} + x^{2(n-1)} + \cdots + x^4 + x^2 + 1.$$

328. 3 (Bui The Hung) Find the smallest possible prime p such that $[(3+\sqrt{p})^{2n}] + 1$ is divisible by 2^{n+1} for each natural number n, where $[x]$ denotes the integral part of x.

328. 4 (Han Ngoc Duc) Find all real numbers a such that there exists a positive real number k and functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the inequality

$$\frac{f(x)+f(y)}{2} \geq f\Big(\frac{x+y}{2}\Big) + k|x-y|^a,$$

for all real numbers x, y.

328. 5 (Vu Hoang Hiep) In space, let $A_1, A_2, \ldots, A_n$ be n distinct points. Prove that

a) $\displaystyle\sum_{i=1}^{n} \angle A_i A_{i+1} A_{i+2} \geq \pi,$

b) $\displaystyle\sum_{i=1}^{n} \angle A_i Q A_{i+1} \leq (n-1)\pi,$

where A_{n+i} is equal to A_i and Q is an arbitrary point distinct from $A_1, A_2, \ldots, A_n$.

329. 1 (Hoang Ngoc Minh) Find the maximum value of the expression

$$(a-b)^4 + (b-c)^4 + (c-a)^4,$$

for any real numbers $1 \leq a, b, c \leq 2$.

331. 1 (Nguyen Manh Tuan) Let x, y, z, w be rational numbers such that $x + y + z + w = 0$. Show that the number

$$\sqrt{(xy - zw)(yz - wx)(zx - yw)}$$

is also rational.

331. 2 (Bui Dinh Than) Given positive reals a, b, c, x, y, z such that

$$a + b + c = 4 \quad \text{and} \quad ax + by + cz = xyz,$$

show that $x + y + z > 4$.

331. 3 (Pham Nang Khanh) Given a triangle ABC and its angle bisector AM, the line perpendicular to BC at M intersects line AB at N. Prove that $\angle BAC$ is a right angle if and only if $MN = MC$.

331. 4 (Dao Tam) Diagonals AC, BD of quadrilateral $ABCD$ intersect at I such that $IA = ID$ and $\angle AID = 120^\circ$. From point M on segment BC, draw $MN \parallel AC$ and $MQ \parallel BD$, N and Q are on AB and CD, respectively. Find the locus of circumcenter of triangle MNQ when M moves on line segment BC.

331. 5 (Nguyen Trong Hiep) Let p, q be primes such that $p > q > 2$. Find all integers k such that the equation $(px - qy)^2 = kxyz$ has integer solutions (x, y, z) with $xy \neq 0$.

331. 6 (Han Ngoc Duc) Let a sequence (u_n), $n = 1, 2, 3, \ldots$, be given defined by $u_n = n^{2^n}$ for all $n = 1, 2, \ldots$. Let

$$x_n = \frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n}.$$

Prove that the sequence (x_n) has a limit as n tends to infinity and that the limit is irrational.

331. 7 (Tran Tuan Anh) Find all positive integers $n \geq 3$ such that the following inequality holds for all real numbers $a_1, a_2, \ldots, a_n$ (assume $a_{n+1} = a_1$)

$$\sum_{1 \leq i < j \leq n} (a_i - a_j)^2 \leq \left(\sum_{i=1}^{n} |a_i - a_{i+1}| \right)^2 .$$

332. 1 (Nguyen Van Ai) Find the remainder in the integer division of the number $a^b + b^a$ by 5, where $a = \overline{22\ldots2}$ with 2004 digits 2, and $b = \overline{33\ldots3}$ with 2005 digits 3 (written in the decimal system).

332. 2 (Nguyen Khanh Nguyen) Suppose that ABC is an isosceles triangle with $AB = AC$. On the line perpendicular to AC at C, let point D such that points B, D are on different sides of AC. Let K be the intersection point of the line perpendicular to AB at B and the line passing through the midpoint M of CD, perpendicular to AD. Compare the lengths of KB and KD.

332. 3 (Pham Van Hoang) Consider the equation

$$x^2 - 2kxy^2 + k(y^3 - 1) = 0,$$

where k is some integer. Prove that the equation has integer solutions (x, y) such that $x > 0$, $y > 0$ if and only if k is a perfect square.

332. 4 (Do Van Ta) Solve the equation

$$\sqrt{x - \sqrt{x - \sqrt{x - \sqrt{x - 5}}}} = 5.$$

332. 5 (Pham Xuan Trinh) Show that if $a \geq 0$ then

$$\sqrt{a} + \sqrt[3]{a} + \sqrt[6]{a} \leq a + 2.$$

332. 6 (Bui Van Chi) Let $ABCD$ be a parallelogram with $AB < BC$. The bisector of angle $\angle BAD$ intersects BC at E; let O be the intersection point of the perpendicular bisectors of BD and CE. A line passing through C parallel to BD intersects the circle with center O and radius OC at F. Determine $\angle AFC$.

332. 7 (Phan Hoang Ninh) Prove that the polynomial

$$p(x) = x^4 - 2003x^3 + (2004 + a)x^2 - 2005x + a$$

with $a \in \mathbb{Z}$ has at most one integer solution. Furthermore, prove that it has no multiple integral root greater than 1.

332. 8 (Phung Van Su) Prove that for any real numbers a, b, c

$$(a^2 + 3)(b^2 + 3)(c^2 + 3) \geq \frac{4}{27}(3ab + 3bc + 3ca + abc)^2.$$

332. 9 (Nguyen Van Thanh) Determine all functions $f(x)$ defined on the interval $(0, +\infty)$ which have a derivative at $x = 1$ and that satisfy

$$f(xy) = \sqrt{x}f(y) + \sqrt{y}f(x)$$

for all positive real numbers x, y.

332. 10 (Hoang Ngoc Canh) Let $A_1A_2 \dots A_n$ be a n-gon inscribed in the unit circle; let M be a point on the minor arc A_1A_n. Prove that

a) $MA_1 + MA_3 + \dots + MA_{n-2} + MA_n < \dfrac{n}{\sqrt{2}}$ for n odd;

b) $MA_1 + MA_3 + \dots + MA_{n-3} + MA_{n-1} \leq \dfrac{n}{\sqrt{2}}$ for n even.

When does equality hold?

332. 11 (Dang Thanh Hai) Let ABC be an equilateral triangle with centroid O; ℓ is a line perpendicular to the plane (ABC) at O. For each point S on ℓ, distinct from O, a pyramid $SABC$ is defined. Let ϕ be the dihedral angle between a lateral face and the base, let γ be the angle between two adjacent lateral faces of the pyramid. Prove that the quantity $F(\phi,\gamma) = \tan^2\phi\,[3\tan^2(\gamma/2) - 1]$ is independent of the position of S on ℓ.

334. 1 (Dang Nhu Tuan) Determine the sum

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{(n-1)n(n+1)} + \cdots + \frac{1}{23\cdot 24\cdot 25}.$$

334. 2 (Nguyen Phuoc) Let ABC be a triangle with angle A not being right, $B \neq 135^\circ$. Let M be the midpoint of BC. A right isosceles triangle ABD is outwardly erected on the side BC as base. Let E be the intersection point of the line through A perpendicular to AB and the line through C parallel to MD. Let AB intersect CE and DM at P and Q, respectively. Prove that Q is the midpoint of BP.

334. 3 (Nguyen Duy Lien) Find the smallest possible odd natural number n such that n^2 can be expressed as the sum of an odd number of consecutive perfect squares.

334. 4 (Pham Viet Hai) Find all positive numbers a, b, c, d such that

$$\frac{a^2}{b+c} + \frac{b^2}{c+d} + \frac{c^2}{d+a} + \frac{d^2}{a+b} = 1 \quad \text{and}$$
$$a^2 + b^2 + c^2 + d^2 \geq 1.$$

334. 5 (Dao Quoc Dung) The incircle of triangle ABC (incenter I) touches the sides BC, CA, and AB respectively at D, E, F. The line through A perpendicular to IA intersects lines DE, DF at M, N, respectively; the line through B perpendicular to IB intersect EF, ED at P, Q, respectively; the line through C perpendicular to IC intersect lines FD, FE at S, T, respectively. Prove the inequality

$$MN + PQ + ST \geq AB + BC + CA.$$

334. 6 (Vu Huu Binh) Let ABC be a right isosceles triangle with $A = 90^\circ$. Find the locus of points M such that $MB^2 - MC^2 = 2MA^2$.

334. 7 (Tran Tuan Anh) We are given n distinct positive numbers, $n \geq 4$. Prove that it is possible to choose at least two numbers such that their sums and differences do not coincide with any $n-2$ others of the given numbers.

335. 1 (Vu Tien Viet) Prove that for all triangles ABC

$$\cos A+\cos B+\cos C\leq 1+\frac{1}{6}\Big(\cos^2\frac{A-B}{2}+\cos^2\frac{B-C}{2}+\cos^2\frac{C-A}{2}\Big).$$

335. 2 (Phan Duc Tuan) In triangle ABC, let $BC=a$, $CA=b$, $AB=c$ and F be its area. Suppose that M,N, and P are points on the sides BC,CA, and AB, respectively. Prove that

$$ab\cdot MN^2+bc\cdot NP^2+ca\cdot PM^2\geq 4F^2.$$

335. 3 (Tran Van Xuan) In isosceles triangle ABC, $\angle ABC=120^\circ$. Let D be the point of intersection of line BC and the tangent to the circumcircle of triangle ABC at A. A line through D and the circumcenter O intersects AB and AC at E and F, respectively. Let M and N be the midpoints of AB and AC. Show that AO, MF and NE are concurrent.

336. 1 (Nguyen Hoa) Solve the following system of equations

$$\begin{aligned}&\frac{a}{x}-\frac{b}{z}=c-zx,\\&\frac{b}{y}-\frac{c}{x}=a-xy,\\&\frac{c}{z}-\frac{a}{y}=b-yz.\end{aligned}$$

336. 2 (Pham Van Thuan) Given two positive real numbers a,b such that $a^2+b^2=1$, prove that

$$\frac{1}{a}+\frac{1}{b}\geq 2\sqrt{2}+\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2.$$

336. 3 (Nguyen Hong Thanh) Let P be an arbitrary point in the interior of triangle ABC. Let $BC=a$, $CA=b$, $AB=c$. Denote by u, v and w the distances of P to the lines BC, CA, AB, respectively. Determine P such that the product uvw is a maximum and calculate this maximum in terms of a,b,c.

336. 4 (Nguyen Lam Tuyen) Given the polynomial $Q(x)=(p-1)x^p-x-1$ with p being an odd prime number. Prove that there exist infinitely many positive integers a such that $Q(a)$ is divisible by p^p.

336. 5 (Hoang Minh Dung) Prove that in any triangle ABC the following inequalities hold:

a) $\cos A+\cos B+\cos C+\cot A+\cot B+\cot C\geq\dfrac{3}{2}+\sqrt{3};$

b) $\sqrt{3}\,(\cos A+\cos B+\cos C)+\cot\dfrac{A}{2}+\cot\dfrac{B}{2}+\cot\dfrac{C}{2}\geq\dfrac{9\sqrt{3}}{2}.$

337. 1 (Nguyen Thi Loan) Given four real numbers a, b, c, d such that $4a^2+b^2 = 2$ and $c+d = 4$, determine the maximum value of the expression $f = 2ac + bd + cd$.

337. 2 (Vu Anh Nam) In triangle ABC, let D be the intersection point of the internal angle bisectors BM and CN, M on AC and N on AB. Prove that $\angle BAC = 90^\circ$ if and only if $2BD \cdot CD = BM \cdot CN$.

337. 3 (Tran Tuan Anh) Determine the maximum value of the expression $f = (x-y)(y-z)(z-x)(x+y+z)$, where x, y, z lie in the interval $[0, 1]$.

337. 4 (Han Ngoc Duc) Let n, $n \geq 2$, be a natural number, a, b be positive real numbers such that $a < b$. Suppose that $x_1, x_2, \ldots, x_n$ are n real numbers in the interval $[a, b]$. Find the maximum value of the sum

$$\sum_{1 \leq i < j \leq n} (x_i - x_j)^2.$$

337. 5 (Le Hoai Bac) A line through the incenter of a triangle ABC intersects sides AB and AC at M and N, respectively. Show that

$$\frac{MB \cdot NC}{MA \cdot NA} \leq \frac{BC^2}{4\,AB \cdot AC}.$$

338. 1 (Pham Thinh) Show that if a, b, c, d, p, q are positive real numbers with $p \geq q$ then the following inequality holds:

$$\frac{a}{pb+qc} + \frac{b}{pc+qd} + \frac{c}{pd+qa} + \frac{d}{pa+qb} \geq \frac{4}{p+q}.$$

Is the inequality still true if $p < q$?

338. 2 (Tran Quang Vinh) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition $f(x^2 + f(y)) = y + xf(x)$ for all real numbers x, y.

338. 3 (Tran Viet Anh) Determine the smallest possible positive integer n such that there exists a polynomial $p(x)$ of degree n with integer coefficients satisfying the conditions

a) $p(0) = 1,\ p(1) = 1$;

b) $p(m)$ divided by 2003 leaves remainders 0 or 1 for all integers $m > 0$.

338. 4 (Hoang Trong Hao) The Fibonacci sequence (F_n), $n = 1, 2, \dots$, is defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n = 2, 3, 4, \dots$. Show that if $a \neq F_{n+1}/F_n$ for all $n = 1, 2, 3, \dots$ then the sequence (x_n), where

$$x_1 = a, \quad x_{n+1} = \frac{1}{1 + x_n}, \quad n = 1, 2, \dots$$

is defined and has a finite limit when n tends to infinity. Determine the limit.

339. 1 (Ngo Van Khuong) Given five positive real numbers a, b, c, d, e such that $a^2 + b^2 + c^2 + d^2 + e^2 \leq 1$, prove that

$$\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + cd} + \frac{1}{1 + de} + \frac{1}{1 + ea} \geq \frac{25}{6}.$$

339. 2 (Le Chu Bien) Suppose that $ABCD$ is a rectangle. The line perpendicular to AC at C intersects lines AB, AD respectively at E, F. Prove the identity $BE\sqrt{CF} + DF\sqrt{CE} = AC\sqrt{EF}$.

339. 3 (Tran Hong Son) Let I be the incenter of triangle ABC and let m_a, m_b, m_c be the lengths of the medians from vertices A, B and C, respectively. Prove that

$$\frac{IA^2}{m_a^2} + \frac{IB^2}{m_b^2} + \frac{IC^2}{m_c^2} \leq \frac{3}{4}.$$

339. 4 (Quach Van Giang) Given three positive real numbers a, b, c such that $ab + bc + ca = 1$. Prove that the minimum value of the expression $x^2 + ry^2 + tz^2$ is $2m$, where m is the root of the cubic equation $2x^3 + (r + s + 1)x^2 - rs = 0$ in the interval $(0, \sqrt{rs})$. Find all primes r, s such that $2m$ is rational.

339. 5 (Nguyen Truong Phong) The sequence (x_n) is defined by

$$x_n = a_n^{a_n}, \quad \text{where} \quad a_n = \frac{(2n)!}{(n!)^2 \cdot 2^{2n}}, \quad \text{for } n = 1, 2, 3, \dots.$$

Prove that the sequence (x_n) has a limit when n tends to infinity and determine the limit.

339. 6 (Huynh Tan Chau) Let a be a real number, $a \in (0, 1)$. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at $x = 0$ and satisfy the equation

$$f(x) - 2f(ax) + f(a^2x) = x^2$$

for all real x.

339. 7 (Nguyen Xuan Hung) In the plane, given a circle with center O and radius r. Let P be a fixed point inside the circle such that $OP = d > 0$. The chords AB and CD through P make a fixed angle α, $(0^\circ < \alpha \leq 90^\circ)$. Find the maximum and minimum value of the sum $AB + CD$ when both AB and CD vary, and determine the position of the two chords.

340. 1 (Pham Hoang Ha) Find the maximum value of the expression

$$\frac{x+y}{1+z} + \frac{y+z}{1+x} + \frac{z+x}{1+y},$$

where x, y, z are real numbers in the interval $[\frac{1}{2}, 1]$.

340. 2 (Nguyen Quynh) Let M be a point interior to triangle ABC, let AM intersect BC at E, let CM meet AB at F. Suppose that N is the reflection of B across the midpoint of EF. Prove that the line MN has a fixed point when M moves in the triangle ABC.

340. 3 (Tran Tuan Anh) Let a, b, c be the side lengths of a triangle, and F its area, prove that $F \leq \frac{\sqrt{3}}{4}(abc)^{2/3}$, and determine equality cases.

340. 4 (Han Ngoc Duc) Given non-negative integers n, k, $n > 1$ and let $\{a_1, a_2, \ldots, a_n\}$ be the n real numbers, prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{\binom{k+2}{k+i+j}} \geq 0.$$

340. 5 (Tran Minh Hien) Does there exist a function $f : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$f^2(x) \geq f(x+y)(f(x)+y)$$

for all positive real numbers x, y?

. . . to be continued