Geometric Constructions and Proofs via Motions

P 1 (e) Given a triangle ABC. Construct a square $PQRS$ such that $\overline{PQ}$ lies on the side $\overline{AB}$, R lies on $\overline{BC}$, and S lies on $\overline{CA}$.

P 2 (e) Given a square $ABCD$. Construct a circle k through D touching the sides AB and BC.

P 3 (e) Given two lines g and h and a circle k. Find a square $ABCD$ with $A, C \in h$, $D \in g$ and $B \in k$.

P 4 (e) Given three parallel lines g, h, and k. Find an equilateral triangle ABC with $A \in g$, $B \in h$, and $C \in k$.

P 5 (e) Given two circles k_1 and k_2 intersecting in S and T. Find two different points $P \in k_1$ and $Q \in k_2$ both different from S and T such that P, Q, and S are collinear and $\left|\overline{PS}\right| = \left|\overline{QS}\right|$.

P 6 (m) Given three concentric circles k_1, k_2, and k_3 with radii 3, 4, and 5, respectively. Draw a right triangle ABC with $\alpha = 60°$, $\beta = 30°$ and $\gamma = 90°$ such that A, B, and C lie on k_1, k_3, and k_2, respectively.

P 7 (m) (a) Given a point P. Construct a square $ABCD$ such that the distances of P from A, B, and C are 1, 2, and 3, respectively. (b) Given a square $ABCD$. Construct a point P such that $\left|\overline{PA}\right| : \left|\overline{PB}\right| : \left|\overline{PC}\right| = 1 : 2 : 3$.

P 8 (e) Given three adjacent congruent squares. Let α be the smallest angle in the right triangle with legs 1 and 2. Let β be the smallest angle in the right triangle with legs 1 and 3. Prove that $\alpha + \beta = 45°$.

P 9 (e) Given a square $ABCD$. Suppose that P divides side $\overline{AC}$ as $4 : 1$ and Q divides $\overline{AB}$ as $3 : 2$. Prove that PQD is a right isosceles triangle.

P 10 (e) Let ABC be an acute angle triangle. Over $\overline{BC}$ and $\overline{AC}$ draw the squares $BPQC$ and $CRSA$ to the outside, respectively. Let M be the midpoint of $\overline{AB}$.
a) Prove that MC and QR are perpendicular.
b) Prove that $\left|\overline{QR}\right| = 2\left|\overline{MC}\right|$.

P 11 (e) Given a right triangle ABC with right angle at C. Let squares $ABHF$, $BCLE$, and $ACKD$ be drawn to the outside. Moreover, let $\triangle HFG$ be congruent to $\triangle ABC$ with the same orientation. Prove that $\overline{CG}$ and $\overline{ED}$ have equal length and are orthogonal.

P 12 (m) Let the points A, B, D, and C be on a line in that order. Draw isosceles right triangles APB and BQC over the hypotenuses $\overline{AB}$ and $\overline{BC}$ downwards, respectively, as well as isosceles right triangles ASD and DRC over the hypotenuses $\overline{AD}$ and $\overline{DC}$ upwards, respectively. Prove that the segments $\overline{SQ}$ and $\overline{PR}$ are perpendicular and of equal length.

P 13 (e) Over the sides $\overline{AB}$ and $\overline{DA}$ of a parallelogram $ABCD$ we draw to the outside of $ABCD$ equilateral triangles $\triangle ABQ$ and $\triangle ADP$, respectively. Prove that $\triangle PQC$ is equilateral.

P 14 (e) Suppose that $\triangle ABC$ is equilateral. Prove that for all points X on the arc $\overset{\frown}{AB}$ of the circumcircle of $\triangle ABC$ we have $\left|\overline{AX}\right| + \left|\overline{BX}\right| = \left|\overline{CX}\right|$.

P 15 (m, BWM 1998.1.3) Over the sides $\overline{BC}$ and $\overline{CA}$ of an acute angle triangle ABC there are drawn to the outside right isosceles triangles BXC and CYA with right angles at X and Y, respectively. Let M be the midpoint of $\overline{AB}$. Prove that XMY is a right isosceles triangle.

P 16 (e, MO 471313) The circles k_1 and k_2 with centers M_1 and M_2, respectively, intersect in different points A and B. The line AM_1 intersects k_1 in A and C. The line AM_2 intersects k_2 in A and in D. Prove that M_1M_2 and CD are parallel lines and B lies on CD.

P 17 (m, Baltic Way 1992) Given a circle k and two circles k_1 and k_2 touching k from the inside at A and B, respectively. Let t be one common tangent line to k_1 and k_2 such that k_1 and k_2 are on the same side of t and let C and D the touching points, respectively. Prove that the lines AC and BD intersect in a point F on k.

P 18 (h, BWM 1996.2.3) Let ABC a triangle. Draw to the outside three rectangles ABB_1A_1, BCC_1B_2, and CAA_2C_2. Prove that the perpendicular bisectors of $\overline{A_1A_2}$, $\overline{B_1B_2}$, and $\overline{C_1C_2}$ meet in one point.

P 19 (h, MO 421142) In the inside of triangle ABC there are four congruent circles k_1, k_2, k_3, and k_4 with centers A', B', C', and M', respectively. The circles k_1, k_2, and k_3 are touching two sides of the triangle each as well as circle k_4 from outside. Prove that M' lies on the line through the centers of circumcircle and incircle of $\triangle ABC$.

P 20 (m) Suppose that three congruent circles k_1, k_2, and k_3 meet in one common point M. Besides in M, the circles intersect in A, B, and C. Prove that the circumcircle k_4 of $\triangle ABC$ is congruent to the given three circles.

P 21 (m) Given three incongruent and non-intersecting circles k_1, k_2, and k_3. To the pair of circles k_2 and k_3 draw the two outer common tangent lines which intersect in Z_1. The intersection points Z_2 and Z_3 are constructed in the same way. Prove that Z_1, Z_2, and Z_3 are collinear.

P 22 (h, IMO 1979.3) Two circles k_1 and k_2 with centers M_1 and M_2 intersect in P. Starting simultaneously from P two points R_1 and R_2 move with constant speed, each travelling along its own circle in the same sense. The two points return to P simultaneously after one revolution. Prove that there is a fixed point F in the plane such that the two points R_1 and R_2 are always equidistant from F.

P 23 (m) Let P be an inner point of the parallelogram $ABCD$ such that $\angle APB + \angle CPD = 180°$. Prove that $\angle PAB = \angle PCB$.

P 24 (e) Let $ABCD$ be a tetrahedron with midpoints E, F, G, and H of the sides $\overline{BD}$, $\overline{CD}$, $\overline{CA}$, and $\overline{AB}$. Prove that $EFGH$ is a parallelogram.

P 25 (e, MO 440923) Let $ABCDE$ be a pyramid with a square basis $ABCD$. The faces $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$ are all equilateral. Over the face $\triangle CDE$ there is drawn to the outside a regular tetrahedron $CDEF$. How many faces has the polyeder $ABCDEF$?

P 26 (e, Bay Area Mathematical Olympiad 1999) Let $ABCD$ be a trapezoid. Over the parallel sides $\overline{AB}$ and $\overline{CD}$ are drawn squares with midpoints P_1 and P_2, respectively. Let P be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$. Prove that P, P_1, and P_2 are collinear.

P 27 (m, BWM 2005.2) Let k_1 and k_2 be circles intersecting in A and B. A first line through B intersects k_1 in C and k_2 in E. A second line through B intersects k_1 in D and k_2 in F. We assume that B lies between C and E and between D and F. Let M and N be the midpoints of $\overline{CE}$ and $\overline{DF}$, respectively. Prove that the triangles ACD, AEF, and AMN are similar to each other.

P 28 (m) Over the sides $\overline{BC}$ and $\overline{CA}$ of an acute triangle ABC there are drawn similar to each other right triangles BXC and AYC with right angles at X and Y. We assume that $\triangle BXC$ and $\triangle AYC$ have opposite orientation. Let M be the midpoint of $\overline{AB}$. Prove that $\triangle XMY$ is isosceles.

P 29 (m, Napoleon) Over the sides of a triangle ABC are drawn to the outside equilateral triangles. Prove that the midpoints of these triangles form an equilateral triangle.

P 30 (h) Over the sides of an acute triangle ABC there are drawn to the outside triangles BCX, CAY, and ABZ, respectively such that $\angle ZAB = \angle ZBA = 15°$, $\angle YAC = \angle XBC = 45°$, and $\angle YCA = \angle XCB = 30°$. Prove that XYZ is a right isosceles triangle.

P 31 (h) Suppose that the vertices of a regular n-gon are on the $\mathbb{Z} \times \mathbb{Z}$ lattice, that is, all coordinates are integers. Prove that $n = 4$.

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