Mathscope is a free problem resource selected from problem solving journals in Vietnam. This freely accessible collection is our effort to introduce elementary mathematics problems to our foreign friends for either recreational or professional use. We would like to give you a new taste of Vietnamese mathematical culture. Whatever the purpose, we welcome suggestions and comments from you all. More communications can be addressed to Pham Van Thuan, 4E2, 565 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam, or email us at pvthuan@vnu.edu.vn.

It’s now not too hard to find problems and solutions on the Internet due to the increasing numbers of websites devoted to mathematical problems solving. Anyway, we hope that this complete collection saves you considerable time searching the problems you really want. We intend to give an outline of solutions to the problems, but it would take time. Now enjoy these “cakes” from Vietnam first.

261.1 (Ho Quang Vinh) Given a triangle $ABC$, its internal angle bisectors $BE$ and $CF$, and let $M$ be any point on the line segment $EF$. Denote by $S_A$, $S_B$, and $S_C$ the areas of triangles $MBC$, $MCA$, and $MAB$, respectively. Prove that

$$\frac{\sqrt{S_B} + \sqrt{S_C}}{\sqrt{S_A}} \leq \sqrt{\frac{AC + AB}{BC}},$$

and determine when equality holds.
261.2 (Editorial Board) Find the maximum value of the expression
\[ A = 13\sqrt{x^2 - x^4} + 9\sqrt{x^2 + x^4} \]
for \(0 \leq x \leq 1\).

261.3 (Editorial Board) The sequence \((a_n), n = 1, 2, 3, \ldots\), is defined by \(a_1 > 0\), and \(a_{n+1} = ca_n^2 + a_n\) for \(n = 1, 2, 3, \ldots\), where \(c\) is a constant. Prove that
\[ a_n \geq \sqrt{cn - 1}n a_1^{n+1}, \]
and
\[ a_1 + a_2 + \cdots + a_n > n\left(na_1 - \frac{1}{c}\right) \]
for \(n \in \mathbb{N}\).

261.4 (Editorial Board) Let \(X, Y, Z\) be the reflections of \(A, B,\) and \(C\) across the lines \(BC, CA,\) and \(AB\), respectively. Prove that \(X, Y,\) and \(Z\) are collinear if and only if
\[ \cos A \cos B \cos C = -\frac{3}{8}. \]

261.5 (Vinh Competition) Prove that if \(x, y, z > 0\) and \(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1\) then the following inequality holds:
\[ \left(1 - \frac{1}{1 + x^2}\right)\left(1 - \frac{1}{1 + y^2}\right)\left(1 - \frac{1}{1 + z^2}\right) > \frac{1}{2}. \]

261.6 (Do Van Duc) Given four real numbers \(x_1, x_2, x_3, x_4\) such that \(x_1 + x_2 + x_3 + x_4 = 0\) and \(|x_1| + |x_2| + |x_3| + |x_4| = 1\), find the maximum value of
\[ \prod_{1 \leq i < j \leq 4} (x_i - x_j). \]

261.7 (Doan Quang Manh) Given a rational number \(x \geq 1\) such that there exists a sequence of integers \((a_n), n = 0, 1, 2, \ldots\), and a constant \(c \neq 0\) such that \(\lim_{n \to \infty} (cx^n - a_n) = 0\). Prove that \(x\) is an integer.

262.1 (Ngo Van Hiep) Let \(ABC\) an equilateral triangle of side length \(a\). For each point \(M\) in the interior of the triangle, choose points \(D, E, F\) on the sides \(CA, AB,\) and \(BC\), respectively, such that \(DE = MA, EF = MB,\) and \(FD = MC\). Determine \(M\) such that \(\triangle DEF\) has smallest possible area and calculate this area in terms of \(a\).

262.2 (Nguyen Xuan Hung) Given is an acute triangle with altitude \(AH\). Let \(D\) be any point on the line segment \(AH\) not coinciding with the endpoints of this segment and the orthocenter of triangle \(ABC\). Let ray \(BD\) intersect \(AC\) at \(M\), ray \(CD\) meet \(AB\) at \(N\). The line perpendicular to \(BM\) at \(M\) meets the line perpendicular to \(CN\) at \(N\) in the point \(S\). Prove that \(\triangle ABC\) is isosceles with base \(BC\) if and only if \(S\) is on line \(AH\).
262. 3 (Nguyen Duy Lien) The sequence \((a_n)\) is defined by
\[ a_0 = 2, \quad a_{n+1} = 4a_n + \sqrt{15a_n^2 - 60} \quad \text{for } n \in \mathbb{N}. \]
Find the general term \(a_n\). Prove that \(\frac{1}{5}(a_{2n} + 8)\) can be expressed as the sum of squares of three consecutive integers for \(n \geq 1\).

262. 4 (Tuan Anh) Let \(p\) be a prime, \(n \) and \(k \) positive integers with \(k > 1\). Suppose that \(b_i, i = 1, 2, \ldots, k, \) are integers such that
i) \(0 \leq b_i \leq k - 1\) for all \(i,\)
ii) \(p^{nk-1}\) is a divisor of \(\left(\sum_{i=1}^{k} p^{nb_i}\right) - p^{n(k-1)} - p^{n(k-2)} - \cdots - p^n - 1.\)
Prove that the sequence \((b_1, b_2, \ldots, b_k)\) is a permutation of the sequence \((0, 1, \ldots, k - 1)\).

262. 5 (Doan The Phiet) Without use of any calculator, determine
\[
\sin \frac{\pi}{14} + 6 \sin^2 \frac{\pi}{14} - 8 \sin^4 \frac{\pi}{14}.
\]

264. 1 (Tran Duy Hinh) Prove that the sum of all squares of the divisors of a natural number \(n\) is less than \(n^2 \sqrt{n}\).

264. 2 (Hoang Ngoc Canh) Given two polynomials
\[ f(x) = x^4 - (1 + e^x) + e^2, \quad g(x) = x^4 - 1, \]
prove that for distinct positive numbers \(a, b\) satisfying \(a^b = b^a\), we have \(f(a)f(b) < 0\) and \(g(a)g(b) > 0\).

264. 3 (Nguyen Phu Yen) Solve the equation
\[
\frac{(x-1)^4}{(x^2-3)^2} + \frac{(x^2-3)^4}{(x-1)^2} + \frac{1}{(x-1)^2} = 3x^2 - 2x - 5.
\]

264. 4 (Nguyen Minh Phuong, Nguyen Xuan Hung) Let \(I\) be the incenter of triangle \(ABC\). Rays \(AI, BI,\) and \(CI\) meet the circumcircle of triangle \(ABC\) again at \(X, Y,\) and \(Z,\) respectively. Prove that
a) \(IX + IY + IZ \geq IA + IB + IC,\)
b) \(\frac{1}{IX} + \frac{1}{IY} + \frac{1}{IZ} \geq \frac{3}{R}\).

265. 1 (Vu Dinh Hoa) The lengths of the four sides of a convex quadrilateral are natural numbers such that the sum of any three of them is divisible by the fourth number. Prove that the quadrilateral has two equal sides.
265. 2 (Dam Van Nhi) Let $AD$, $BE$, and $CF$ be the internal angle bisectors of triangle $ABC$. Prove that $p(DEF) \leq \frac{1}{2}p(ABC)$, where $p(XYZ)$ denotes the perimeter of triangle $XYZ$. When does equality hold?

266. 1 (Le Quang Nam) Given real numbers $x, y, z \geq -1$ satisfying $x^3 + y^3 + z^3 \geq x^2 + y^2 + z^2$, prove that $x^5 + y^5 + z^5 \geq x^2 + y^2 + z^2$.

266. 2 (Dang Nhon) Let $ABCD$ be a rhombus with $\angle A = 120^\circ$. A ray $Ax$ and $AB$ make an angle of $15^\circ$, and $Ax$ meets $BC$ and $CD$ at $M$ and $N$, respectively. Prove that $\frac{3}{AM^2} + \frac{3}{AN^2} = \frac{4}{AB^2}$.

266. 3 (Ha Duy Hung) Given an isosceles triangle with $\angle A = 90^\circ$. Let $M$ be a variable point on line $BC$, ($M$ distinct from $B$, $C$). Let $H$ and $K$ be the orthogonal projections of $M$ onto lines $AB$ and $AC$, respectively. Suppose that $I$ is the intersection of lines $CH$ and $BK$. Prove that the line $MI$ has a fixed point.

266. 4 (Luu Xuan Tinh) Let $x, y$ be real numbers in the interval $(0, 1)$ and $x + y = 1$, find the minimum of the expression $x^x + y^y$.

267. 1 (Do Thanh Han) Let $x, y, z$ be real numbers such that
\[ x^2 + z^2 = 1, \]
\[ y^2 + 2y(x + z) = 6. \]
Prove that $y(z - x) \leq 4$, and determine when equality holds.

267. 2 (Le Quoc Han) In triangle $ABC$, medians $AM$ and $CN$ meet at $G$. Prove that the quadrilateral $BMGN$ has an incircle if and only if triangle $ABC$ is isosceles at $B$.

267. 3 (Tran Nam Dung) In triangle $ABC$, denote by $a, b, c$ the side lengths, and $F$ the area. Prove that
\[ F \leq \frac{1}{16}(3a^2 + 2b^2 + 2c^2), \]
and determine when equality holds. Can we find another set of the coefficients of $a^2$, $b^2$, and $c^2$ for which equality holds?

268. 1 (Do Kim Son) In a triangle, denote by $a, b, c$ the side lengths, and let $r, R$ be the inradius and circumradius, respectively. Prove that
\[ a(b + c - a)^2 + b(c + a - b)^2 + c(a + b - c)^2 \leq 6\sqrt{3}R^2(2R - r). \]
268.2 (Dang Hung Thang) The sequence \((a_n), n \in \mathbb{N}\), is defined by
\[
a_0 = a, \quad a_1 = b, \quad a_{n+2} = da_{n+1} - a_n \quad \text{for} \quad n = 0, 1, 2, \ldots,
\]
where \(a, b\) are non-zero integers, \(d\) is a real number. Find all \(d\) such that \(a_n\) is an integer for \(n = 0, 1, 2, \ldots\).

271.1 (Doan The Phiet) Find necessary and sufficient conditions with respect to \(m\) such that the system of equations
\[
\begin{align*}
x^2 + y^2 + z^2 + xy - yz - zx &= 1, \\
y^2 + z^2 + yz &= 2, \\
z^2 + x^2 + zx &= m
\end{align*}
\]
has a solution.

272.1 (Nguyen Xuan Hung) Given are three externally tangent circles \((O_1), (O_2),\) and \((O_3)\). Let \(A, B, C\) be respectively the points of tangency of \((O_1)\) and \((O_3)\), \((O_2)\) and \((O_3)\), \((O_1)\) and \((O_2)\). The common tangent of \((O_1)\) and \((O_2)\) meets \(C\) and \((O_3)\) at \(M\) and \(N\). Let \(D\) be the midpoint of \(MN\). Prove that \(C\) is the center of one of the excircles of triangle \(ABD\).

272.2 (Trinh Bang Giang) Let \(ABCD\) be a convex quadrilateral such that \(AB + CD = BC + DA\). Find the locus of points \(M\) interior to quadrilateral \(ABCD\) such that the sum of the distances from \(M\) to \(AB\) and \(CD\) is equal to the sum of the distances from \(M\) to \(BC\) and \(DA\).

272.3 (Ho Quang Vinh) Let \(M\) and \(m\) be the greatest and smallest numbers in the set of positive numbers \(a_1, a_2, \ldots, a_n, n \geq 2\). Prove that
\[
\left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} \frac{1}{a_i} \right) \leq n^2 + \frac{n(n-1)}{2} \left( \sqrt{\frac{M}{m}} - \sqrt{\frac{m}{M}} \right)^2.
\]

272.4 (Nguyen Huu Du) Find all primes \(p\) such that
\[
f(p) = (2 + 3) - (2^2 + 3^2) + (2^3 + 3^3) - \cdots - (2^{p-1} + 3^{p-1}) + (2^p + 3^p)
\]
is divisible by 5.

274.1 (Dao Manh Thang) Let \(p\) be the semiperimeter and \(R\) the circumradius of triangle \(ABC\). Furthermore, let \(D, E, F\) be the excenters. Prove that
\[
DE^2 + EF^2 + FD^2 \geq 8\sqrt{3}pR,
\]
and determine the equality case.
274. 2 (Doan The Phiet) Determine the positive root of the equation
\[ x \ln \left( 1 + \frac{1}{x} \right)^{1 + \frac{1}{x}} - \frac{x^3 \ln \left( 1 + \frac{1}{x^2} \right)^{1 + \frac{1}{x^2}}}{x} = 1 - x. \]

274. 3 (N.Khanh Nguyen) Let \( ABCD \) be a cyclic quadrilateral. Points \( M, N, P, \) and \( Q \) are chosen on the sides \( AB, BC, CD, \) and \( DA, \) respectively, such that \( MA/MB = PD/PC = AD/BC \) and \( QA/QD = NB/NC = AB/CD. \) Prove that \( MP \) is perpendicular to \( NQ. \)

274. 4 (Nguyen Hao Lieu) Prove the inequality for \( x \in \mathbb{R}: \)
\[ 1 + 2x \arctan x \geq 1 + e^{-\frac{x}{2}}. \]

275. 1 (Tran Hong Son) Let \( x, y, z \) be real numbers in the interval \([-2, 2], \) prove the inequality
\[ 2(x^6 + y^6 + z^6) - (x^4y^2 + y^4z^2 + z^4x^2) \leq 192. \]

276. 1 (Vu Duc Canh) Find the maximum value of the expression
\[ f = \frac{a^3 + b^3 + c^3}{abc}, \]
where \( a, b, c \) are real numbers lying in the interval \([1, 2].\)

276. 2 (Ho Quang Vinh) Given a triangle \( ABC \) with sides \( BC = a, \) \( CA = b, \) and \( AB = c. \) Let \( R \) and \( r \) be the circumradius and inradius of the triangle, respectively. Prove that
\[ \frac{a^3 + b^3 + c^3}{abc} \geq 4 - \frac{2r}{R}. \]

276. 3 (Pham Hoang Ha) Given a triangle \( ABC, \) let \( P \) be a point on the side \( BC, \) let \( H, K \) be the orthogonal projections of \( P \) onto \( AB, AC \) respectively. Points \( M, N \) are chosen on \( AB, AC \) such that \( PM \parallel AC \) and \( PN \parallel AB. \) Compare the areas of triangles \( PHK \) and \( PMN. \)

276. 4 (Do Thanh Han) How many 6-digit natural numbers exist with the distinct digits and two arbitrary consecutive digits cannot be simultaneously odd numbers?

277. 1 (Nguyen Hoi) The incircle with center \( O \) of a triangle touches the sides \( AB, AC, \) and \( BC \) respectively at \( D, E, \) and \( F. \) The escribed circle of triangle \( ABC \) in the angle \( A \) has center \( Q \) and touches the side \( BC \) and the rays \( AB, AC \) respectively at \( K, H, \) and \( I. \) The line \( DE \) meets the rays \( BO \) and \( CO \) respectively at \( M \) and \( N. \) The line \( HI \) meets the rays \( BQ \) and \( CQ \) at \( R \) and \( S, \) respectively. Prove that
\[ \text{a) } \triangle FMN = \triangle KRS, \quad \text{b) } \frac{IS}{AB} = \frac{SR}{BC} = \frac{RH}{CA}. \]
277. 2 (Nguyen Duc Huy) Find all rational numbers $p, q, r$ such that
\[ p \cos \frac{\pi}{7} + q \cos \frac{2\pi}{7} + r \cos \frac{3\pi}{7} = 1. \]

277. 3 (Nguyen Xuan Hung) Let $ABCD$ be a bicentric quadrilateral inscribed in a circle with center $I$ and circumscribed about a circle with center $O$. A line through $I$, parallel to a side of $ABCD$, intersects its two opposite sides at $M$ and $N$. Prove that the length of $MN$ does not depend on the choice of side to which the line is parallel.

277. 4 (Dinh Thanh Trung) Let $x \in (0, \pi)$ be real number and suppose that $\frac{x}{\pi}$ is not rational. Define
\[ S_1 = \sin x, \quad S_2 = \sin x + \sin 2x, \quad \ldots, \quad S_n = \sin x + \sin 2x + \cdots + \sin nx. \]
Let $t_n$ be the number of negative terms in the sequence $S_1, S_2, \ldots, S_n$. Prove that $\lim_{n \to \infty} \frac{t_n}{n} = \frac{x}{\pi^2}$.

279. 1 (Nguyen Huu Bang) Find all natural numbers $a > 1$, such that if $p$ is a prime divisor of $a$ then the number of all divisors of $a$ which are relatively prime to $p$, is equal to the number of the divisors of $a$ that are not relatively prime to $p$.

279. 2 (Le Duy Ninh) Prove that for all real numbers $a, b, x, y$ satisfying $x + y = a + b$ and $x^4 + y^4 = a^4 + b^4$ then $x^n + y^n = a^n + b^n$ for all $n \in \mathbb{N}$.

279. 3 (Nguyen Huu Phuoc) Given an equilateral triangle $ABC$, find the locus of points $M$ interior to $ABC$ such that if the orthogonal projections of $M$ onto $BC, CA$ and $AB$ are $D, E$, and $F$, respectively, then $AD, BE$, and $CF$ are concurrent.

279. 4 (Nguyen Minh Ha) Let $M$ be a point in the interior of triangle $ABC$ and let $X, Y, Z$ be the reflections of $M$ across the sides $BC, CA$, and $AB$, respectively. Prove that triangles $ABC$ and $XYZ$ have the same centroid.

279. 5 (Vu Duc Son) Find all positive integers $n$ such that $n < t_n$, where $t_n$ is the number of positive divisors of $n^2$.

279. 6 (Tran Nam Dung) Find the maximum value of the expression
\[ \frac{x}{1 + x^2} + \frac{y}{1 + y^2} + \frac{z}{1 + z^2}, \]
where $x, y, z$ are real numbers satisfying the condition $x + y + z = 1$. 

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279.7 (Hoang Hoa Trai) Given are three concentric circles with center $O$, and radii $r_1 = 1$, $r_2 = \sqrt{2}$, and $r_3 = \sqrt{5}$. Let $A, B, C$ be three non-collinear points lying respectively on these circles and let $F$ be the area of triangle $ABC$. Prove that $F \leq 3$, and determine the side lengths of triangle $ABC$.

281.1 (Nguyen Xuan Hung) Let $P$ be a point exterior to a circle with center $O$. From $P$ construct two tangents touching the circle at $A$ and $B$. Let $Q$ be a point, distinct from $P$, on the circle. The tangent at $Q$ of the circle intersects $AB$ and $AC$ at $E$ and $F$, respectively. Let $BC$ intersect $OE$ and $OF$ at $X$ and $Y$, respectively. Prove that $XY/EF$ is a constant when $P$ varies on the circle.

281.2 (Ho Quang Vinh) In a triangle $ABC$, let $BC = a$, $CA = b$, $AB = c$ be the sides, $r_a$, $r_b$, and $r_c$ be the inradius and exradii. Prove that
\[
\frac{abc}{r} \geq \frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c}.
\]

283.1 (Tran Hong Son) Simplify the expression
\[
\sqrt{x(4-y)(4-z)} + \sqrt{y(4-z)(4-x)} + \sqrt{z(4-x)(4-y)} - \sqrt{xyz},
\]
where $x, y, z$ are positive numbers such that $x + y + z + \sqrt{xyz} = 4$.

283.2 (Nguyen Phuoc) Let $ABCD$ be a convex quadrilateral, $M$ be the midpoint of $AB$. Point $P$ is chosen on the segment $AC$ such that lines $MP$ and $BC$ intersect at $T$. Suppose that $Q$ is on the segment $BD$ such that $BQ/QD = AP/PC$. Prove that the line $TQ$ has a fixed point when $P$ moves on the segment $AC$.

284.1 (Nguyen Huu Bang) Given an integer $n > 0$ and a prime $p > n + 1$, prove or disprove that the following equation has integer solutions:
\[
1 + \frac{x}{n+1} + \frac{x^2}{2n+1} + \cdots + \frac{x^p}{pn+1} = 0.
\]

284.2 (Le Quang Nam) Let $x, y$ be real numbers such that
\[
(x + \sqrt{1+y^2})(y + \sqrt{1+x^2}) = 1,
\]
prove that
\[
(x + \sqrt{1+x^2})(y + \sqrt{1+y^2}) = 1.
\]
284. 3 (Nguyen Xuan Hung) The internal angle bisectors $AD$, $BE$, and $CF$ of a triangle $ABC$ meet at point $Q$. Prove that if the inradii of triangles $AQF$, $BQD$, and $CQE$ are equal then triangle $ABC$ is equilateral.

284. 4 (Tran Nam Dung) Disprove that there exists a polynomial $p(x)$ of degree greater than 1 such that if $p(x)$ is an integer then $p(x + 1)$ is also an integer for $x \in \mathbb{R}$.

285. 1 (Nguyen Duy Lien) Given an odd natural number $p$ and integers $a, b, c, d, e$ such that $a + b + c + d + e$ and $a^2 + b^2 + c^2 + d^2 + e^2$ are all divisible by $p$. Prove that $a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$ is also divisible by $p$.

285. 2 (Vu Duc Canh) Prove that if $x, y \in \mathbb{R}^*$ then
$$\frac{2x^2 + 3y^2}{2x^3 + 3y^3} + \frac{2y^2 + 3x^2}{2y^3 + 3x^3} \leq \frac{4}{x + y}.$$

285. 3 (Nguyen Huu Phuoc) Let $P$ be a point in the interior of triangle $ABC$. Rays $AP$, $BP$, and $CP$ intersect the sides $BC$, $CA$, and $AB$ at $D$, $E$, and $F$, respectively. Let $K$ be the point of intersection of $DE$ and $CM$, $H$ be the point of intersection of $DF$ and $BM$. Prove that $AD$, $BK$ and $CH$ are concurrent.

285. 4 (Tran Tuan Anh) Let $a, b, c$ be non-negative real numbers, determine all real numbers $x$ such that the following inequality holds:
$$[a^2 + b^2 + (x - 1)c^2][a^2 + c^2 + (x - 1)b^2][b^2 + c^2 + (x - 1)a^2] \leq (a^2 +xbc)(b^2 +xac)(c^2 +xab).$$

285. 5 (Truong Cao Dung) Let $O$ and $I$ be the circumcenter and incenter of a triangle $ABC$. Rays $AI$, $BI$, and $CI$ meet the circumcircle at $D$, $E$, and $F$, respectively. Let $R_a$, $R_b$, and $R_c$ be the radii of the escribed circles of $\triangle ABC$, and let $R_d$, $R_e$, and $R_f$ be the radii of the escribed circles of triangle $DEF$. Prove that
$$R_a + R_b + R_c \leq R_d + R_e + R_f.$$

285. 6 (Do Quang Duong) Determine all integers $k$ such that the sequence defined by $a_1 = 1$, $a_{n+1} = 5a_n + \sqrt{ka_n^2 - 8}$ for $n = 1, 2, 3, \ldots$ includes only integers.

286. 1 (Tran Hong Son) Solve the equation
$$18x^2 - 18x\sqrt{x} - 17x - 8\sqrt{x} - 2 = 0.$$
286. 2 (Pham Hung) Let $ABCD$ be a square. Points $E$, $F$ are chosen on $CB$ and $CD$, respectively, such that $BE/BC = k$, and $DF/DC = (1 - k)/(1 + k)$, where $k$ is a given number, $0 < k < 1$. Segment $BD$ meets $AE$ and $AF$ at $H$ and $G$, respectively. The line through $A$, perpendicular to $EF$, intersects $BD$ at $P$. Prove that $PG/PH = DG/BH$.

286. 3 (Vu Dinh Hoa) In a convex hexagon, the segment joining two of its vertices, dividing the hexagon into two quadrilaterals is called a principal diagonal. Prove that in every convex hexagon, in which the length of each side is equal to 1, there exists a principal diagonal with length not greater than 2 and there exists a principal diagonal with length greater than $\sqrt{3}$.

286. 4 (Do Ba Chu) Prove that in any acute or right triangle $ABC$ the following inequality holds:

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2} \geq \frac{10\sqrt{3}}{9}.$$

286. 5 (Tran Tuan Diep) In triangle $ABC$, no angle exceeding $\pi/2$, and each angle is greater than $\pi/4$. Prove that

$$\cot A + \cot B + \cot C + 3 \cot A \cot B \cot C \leq 4(2 - \sqrt{2}).$$

287. 1 (Tran Nam Dung) Suppose that $a, b$ are positive integers such that $2a - 1, 2b - 1$ and $a + b$ are all primes. Prove that $a^b + b^a$ and $a^n + b^n$ are not divisible by $a + b$.

287. 2 (Pham Dinh Truong) Let $ABCD$ be a square in which the two diagonals intersect at $E$. A line through $A$ meets $BC$ at $M$ and intersects $CD$ at $N$. Let $K$ be the intersection point of $EM$ and $BN$. Prove that $CK \perp BN$.

287. 3 (Nguyen Xuan Hung) Let $ABC$ be a right isosceles triangle, $\angle A = 90^\circ$, $I$ be the incenter of the triangle, $M$ be the midpoint of $BC$. Let $MI$ intersect $AB$ at $N$ and $E$ be the midpoint of $IN$. Furthermore, $F$ is chosen on side $BC$ such that $FC = 3FB$. Suppose that the line $EF$ intersects $AB$ and $AC$ at $D$ and $K$, respectively. Prove that $\triangle ADK$ is isosceles.

287. 4 (Hoang Hoa Trai) Given a positive integer $n$, and $w$ is the sum of $n$ first integers. Prove that the equation

$$x^3 + y^3 + z^3 + t^3 = 2w^3 - 1$$

has infinitely many integer solutions.
288. 1 (Vu Duc Canh) Find necessary and sufficient conditions for $a, b, c$ for which the following equation has no solutions:

$$a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c = x.$$ 

288. 2 (Pham Ngoc Quang) Let $ABCD$ be a cyclic quadrilateral, $P$ be a variable point on the arc $BC$ not containing $A$, and $F$ be the foot of the perpendicular from $C$ onto $AB$. Suppose that $\triangle MEF$ is equilateral, calculate $IK/R$, where $I$ is the incenter of triangle $ABC$ and $K$ the intersection (distinct from $A$) of ray $AI$ and the circumcircle of radius $R$ of triangle $ABC$.

288. 3 (Nguyen Van Thong) Given a prime $p > 2$ such that $p - 2$ is divisible by 3. Prove that the set of integers defined by $y^2 - x^3 - 1$, where $x, y$ are non-negative integers smaller than $p$, has at most $p - 1$ elements divisible by $p$.

289. 1 (Thai Nhat Phuong) Let $ABC$ be a right isosceles triangle with $A = 90^\circ$. Let $M$ be the midpoint of $BC$, $G$ be a point on side $AB$ such that $GB = 2GA$. Let $GM$ intersect $CA$ at $D$. The line through $M$, perpendicular to $CG$ at $E$, intersects $AC$ at $K$. Finally, let $P$ be the point of intersection of $DE$ and $GK$. Prove that $DE = BC$ and $PG = PE$.

289. 2 (Ho Quang Vinh) Given a convex quadrilateral $ABCD$, let $M$ and $N$ be the midpoints of $AD$ and $BC$, respectively, $P$ be the point of intersection of $AN$ and $BM$, and $Q$ the intersection point of $DN$ and $CM$. Prove that

$$\frac{PA}{PN} + \frac{PB}{PM} + \frac{QC}{QM} + \frac{QD}{QN} \geq 4,$$

and determine when equality holds.

290. 1 (Nguyen Song Minh) Given $x, y, z, t \in \mathbb{R}$ and real polynomial

$$F(x, y, z, t) = 9(x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2) + 6xyz(y^2 + t^2) - 4xzt.$$

a) Prove that the polynomial can be factored into the product of two quadratic polynomials.

b) Find the minimum value of the polynomial $F$ if $xy + zt = 1$.

290. 2 (Pham Hoang Ha) Let $M$ be a point on the internal angle bisector $AD$ of triangle $ABC$, $M$ distinct from $A, D$. Ray $AM$ intersects side $AC$ at $E$, ray $CM$ meets side $AB$ at $F$. Prove that if

$$\frac{1}{AB^2} + \frac{1}{AE^2} = \frac{1}{AC^2} + \frac{1}{AF^2},$$

then $\triangle ABC$ is isosceles.
290. 3 (Do Anh) Consider a triangle $ABC$ and its incircle. The internal angle bisector $AD$ and median $AM$ intersect the incircle again at $P$ and $Q$, respectively. Compare the lengths of $DP$ and $MQ$.

290. 4 (Nguyen Duy Lien) Find all pairs of integers $(a, b)$ such that $a + b^2$ divides $a^2b - 1$.

290. 5 (Dinh Thanh Trung) Determine all real functions $f(x), g(x)$ such that $f(x) - f(y) = \cos(x + y) \cdot g(x - y)$ for all $x, y \in \mathbb{R}$.

290. 6 (Nguyen Minh Duc) Find all real numbers $a$ such that the system of equations has real solutions in $x, y, z$:

$$\sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1} = a - 1,$$
$$\sqrt{x + 1} + \sqrt{y + 1} + \sqrt{z + 1} = a + 1.$$ 

290. 7 (Doan Kim Sang) Given a positive integer $n$, find the number of positive integers, not exceeding $n(n + 1)(n + 2)$, which are divisible by $n, n + 1$, and $n + 2$.

291. 1 (Bui Minh Duy) Given three distinct numbers $a, b, c$ such that

$$\frac{a}{b - c} + \frac{b}{c - a} + \frac{c}{a - b} = 0,$$
prove that any two of the numbers have different signs.

291. 2 (Do Thanh Han) Given three real numbers $x, y, z$ that satisfy the conditions $0 < x < y \leq z \leq 1$ and $3x + 2y + z \leq 4$. Find the maximum value of the expression $3x^3 + 2y^2 + z^2$.

291. 3 (Vi Quoc Dung) Given a circle of center $O$ and two points $A, B$ on the circle. A variable circle through $A, B$ has center $Q$. Let $P$ be the reflection of $Q$ across the line $AB$. Line $AP$ intersects the circle $O$ again at $E$, while line $BE$, $E$ distinct from $B$, intersects the circle $Q$ again at $F$. Prove that $F$ lies on a fixed line when circle $Q$ varies.

291. 4 (Vu Duc Son) Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(f(x) + y) = x + f(y) \text{ for } x, y \in \mathbb{Q}.$$ 

291. 5 (Nguyen Van Thong) Find the maximum value of the expression

$$x^2(y - z) + y^2(z - y) + z^2(1 - z),$$
where $x, y, z$ are real numbers such that $0 \leq x \leq y \leq z \leq 1.$
291.6 (Vu Thanh Long) Given an acute-angled triangle $ABC$ with side lengths $a, b, c$. Let $R, r$ denote its circumradius and inradius, respectively, and $F$ its area. Prove the inequality

$$ab + bc + ca \geq 2R^2 + 2Rr + \frac{8}{3\sqrt{3}}F.$$

292.1 (Thai Nhat Phuong, Tran Ha) Let $x, y, z$ be positive numbers such that $xyz = 1$, prove the inequality

$$\frac{x^2}{x + y + y^2z} + \frac{y^2}{y + z + z^3x} + \frac{z^2}{z + x + x^3y} \leq 1.$$

292.2 (Pham Ngoc Boi) Let $p$ be an odd prime, let $a_1, a_2, \ldots, a_{p-1}$ be $p - 1$ integers that are not divisible by $p$. Prove that among the sums $T = k_1a_1 + k_2a_2 + \cdots + k_{p-1}a_{p-1}$, where $k_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, p-1$, there exists at least a sum $T$ divisible by $p$.

292.3 (Ha Vu Anh) Given are two circles $\Gamma_1$ and $\Gamma_2$ intersecting at two distinct points $A, B$ and a variable point $P$ on $\Gamma_1$, $P$ distinct from $A$ and $B$. The lines $PA, PB$ intersect $\Gamma_2$ at $D$ and $E$, respectively. Let $M$ be the midpoint of $DE$. Prove that the line $MP$ has a fixed point.

294.1 (Phung Trong Thuc) Triangle $ABC$ is inscribed in a circle of center $O$. Let $M$ be a point on side $AC$, $M$ distinct from $A, C$, the line $BM$ meets the circle again at $N$. Let $Q$ be the intersection of a line through $A$ perpendicular to $AB$ and a line through $N$ perpendicular to $NC$. Prove that the line $QM$ has a fixed point when $M$ varies on $AC$.

294.2 (Tran Xuan Bang) Let $A, B$ be the intersections of circle $O$ of radius $R$ and circle $O'$ of radius $R'$. A line touches circle $O$ and $O'$ at $T$ and $T'$, respectively. Prove that $B$ is the centroid of triangle $ATT'$ if and only if

$$OO' = \frac{\sqrt{3}}{2}(R + R').$$

294.3 (Vu Tri Duc) If $a, b, c$ are positive real numbers such that $ab + bc + ca = 1$, find the minimum value of the expression $w(a^2 + b^2) + c^2$, where $w$ is a positive real number.

294.4 (Le Quang Nam) Let $p$ be a prime greater than 3, prove that $(p - 1)_{2001p^2 - 1} - 1$ is divisible by $p^3$. 

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294. 5 (Truong Ngoc Dac) Let \( x, y, z \) be positive real numbers such that \( x = \max\{x, y, z\} \), find the minimum value of
\[
\frac{x}{y} + \sqrt{1 + \frac{y}{z} + \sqrt{1 + \frac{z}{x}}}
\]

294. 6 (Pham Hoang Ha) The sequence \( (a_n), n = 1, 2, 3, \ldots, \) is defined
by \( a_n = \frac{1}{n^2(n+2)} \) for \( n = 1, 2, 3, \ldots, \)
Prove that
\[
a_1 + a_2 + \cdots + a_n < \frac{1}{2\sqrt{2}} \quad \text{for } n = 1, 2, 3, \ldots.
\]

294. 7 (Vu Huy Hoang) Given are a circle \( O \) of radius \( R \), and an odd natural number \( n \). Find the positions of \( n \) points \( A_1, A_2, \ldots, A_n \) on the circle such that the sum
\[
A_1 A_2 + A_2 A_3 + \cdots + A_{n-1} A_n + A_n A_1
\]
is a minimum.

295. 1 (Tran Tuyet Thanh) Solve the equation
\[
x^2 - x - 1000\sqrt{1 + 8000x} = 1000.
\]

295. 2 (Pham Dinh Truong) Let \( A_1 A_2 A_3 A_4 A_5 A_6 \) be a convex hexagon
with parallel opposite sides. Let \( B_1, B_2, \) and \( B_3 \) be the points of inter-
section of pairs of diagonals \( A_1 A_4 \) and \( A_2 A_5, A_2 A_5 \) and \( A_3 A_6, A_3 A_6 \) and
\( A_1 A_4, \) respectively. Let \( C_1, C_2, C_3 \) be respectively the midpoints of the
segments \( A_3 A_6, A_1 A_4, A_2 A_5 \). Prove that \( B_1 C_1, B_2 C_2, B_3 C_3 \) are concur-
tent.

295. 3 (Bui The Hung) Let \( A, B \) be respectively the greatest and small-
est numbers from the set of \( n \) positive numbers \( x_1, x_2, \ldots, x_n, n \geq 2 \). Prove that
\[
A < \frac{(x_1 + x_2 + \cdots + x_n)^2}{x_1 + 2x_2 + \cdots + nx_n} < 2B.
\]

295. 4 (Tran Tuan Anh) Prove that if \( x, y, z > 0 \) then
a) \( (x + y + z)^3(y + z - x)(z + x - y)(x + y - z) \leq 27x^3 y^3 z^3, \)
b) \( (x^2 + y^2 + z^2)(y + z - x)(z + x - y)(x + y - z) \leq xyz(yz + zx + xy), \)
c) \( (x + y + z) [2(yz + zx + xy) - (x^2 + y^2 + z^2)] \leq 9xyz. \)

295. 5 (Vu Thi Hue Phuong) Find all functions \( f : \mathbb{D} \rightarrow \mathbb{D}, \) where
\( \mathbb{D} = [1, +\infty) \) such that
\[
f(x f(y)) = y f(x) \quad \text{for } x, y \in \mathbb{D}.
\]
295. 6 (Nguyen Viet Long) Given an even natural number \( n \), find all polynomials \( p_n(x) \) of degree \( n \) such that

i) all the coefficients of \( p_n(x) \) are elements from the set \( \{0, -1, 1\} \) and \( p_n(0) \neq 0 \);

ii) there exists a polynomial \( q(x) \) with coefficients from the set \( \{0, -1, 1\} \) such that \( p_n(x) \equiv (x^2 - 1)q(x) \).

296. 1 (Thoi Ngoc Anh) Prove that

\[
\frac{1}{6} < \frac{3 - \sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}}{n \text{ times}} < \frac{5}{2n},
\]

where there are \( n \) radical signs in the expression of the numerator and \( n - 1 \) ones in the expression of the denominator.

296. 2 (Vi Quoc Dung) Let \( ABC \) be a triangle and \( M \) the midpoint of \( BC \). The external angle bisector of \( A \) meets \( BC \) at \( D \). The circumcircle of triangle \( ADM \) intersects line \( AB \) and line \( AC \) at \( E \) and \( F \), respectively. If \( N \) is the midpoint of \( EF \), prove that \( MN \parallel AD \).

296. 3 (Nguyen Van Hien) Let \( k, n \in \mathbb{N} \) such that \( k < n \). Prove that

\[
\frac{(n + 1)^{n+1}}{(k + 1)^{k+1}(n - k + 1)^{n-k+1}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k+1)^{n-k}}.
\]

297. 1 (Nguyen Huu Phuoc) Given a circle with center \( O \) and diameter \( EF \). Points \( N, P \) are chosen on line \( EF \) such that \( ON = OP \). From a point \( M \) interior to the circle, not lying on \( EF \), draw \( MN \) intersecting the circle at \( A \) and \( C \), draw \( MP \) meeting the circle at \( B \) and \( D \) such that \( B \) and \( O \) are on different sides of \( AC \). Let \( K \) be the point of intersection of \( OB \) and \( AC \), \( Q \) the point of intersection of \( EF \) and \( CD \). Prove that lines \( KQ, BD, AO \) are concurrent.

297. 2 (Tran Nam Dung) Let \( a \) and \( b \) two relatively prime numbers. Prove that there exist exactly \( \frac{1}{2}(ab - a - b + 1) \) natural numbers that can not be written in the form \( ax + by \), where \( x \) and \( y \) are non-negative integers.
297. 3 (Le Quoc Han) The circle with center $I$ and radius $r$ touches the sides $BC = a$, $CA = b$, and $AB = c$ of triangle $ABC$ at $M, N,$ and $P,$ respectively. Let $F$ be the area of triangle $ABC$ and $h_a, h_b, h_c$ be the lengths of the altitudes of $\triangle ABC$. Prove that
\begin{align*}
a) \quad 4F^2 &= ab \cdot MN^2 + bc \cdot NP^2 + ca \cdot PM^2; \\
b) \quad \frac{MN^2}{h_a h_b} + \frac{NP^2}{h_b h_c} + \frac{PM^2}{h_c h_a} &= 1.
\end{align*}

298. 1 (Pham Hoang Ha) Let $P$ be the midpoint of side $BC$ of triangle $ABC$ and let $BE, CF$ be two altitudes of the triangle. The line through $A$, perpendicular to $PF$, meets $CF$ at $M$; the line through $A$, perpendicular to $PE$, intersects $BE$ at $N$. Let $K$ and $G$ be respectively the midpoints of $BM$ and $CN$. Finally, let $H$ be the intersection of $KF$ and $GE$. Prove that $AH$ is perpendicular to $EF$.

298. 2 (Pham Dinh Truong) Let $ABCD$ be a square. Points $E$ and $F$ are chosen on sides $AB$ and $CD$, respectively, such that $AE = CF$. Let $AD$ intersect $CE$ and $BF$ at $M$ and $N$, respectively. Suppose that $P$ is the intersection of $BM$ and $CN$, find the locus of $P$ when $E$ and $F$ move on the side $AB$ and $CD$, respectively.

298. 3 (Nguyen Minh Ha) Let $ABCD$ be a convex quadrilateral, let $AB$ intersect $CD$ at $E$; $AD$ intersects $BC$ at $F$. Prove that the midpoints of line segments $AB$, $CD$, and $EF$ are collinear.

298. 4 (Nguyen Minh Ha) Given a cyclic quadrilateral $ABCD$, $M$ is any point in the plane. Let $X, Y, Z, T, U, V$ be the orthogonal projections of $M$ on the lines $AB, CD, AC, DB, AD,$ and $BC$. Let $E, F, G$ be the midpoints of $XY, ZT,$ and $UV$. Prove that $E, F,$ and $G$ are collinear.

300. 1 (Vu Tri Duc) Find the maximum and minimum values of the expression $x \sqrt{1+y} + y \sqrt{1+x}$, where $x, y$ are non-negative real numbers such that $x + y = 1$.

300. 2 (Nguyen Xuan Hung) Let $P$ be a point in the interior of triangle $ABC$. The incircle of triangle $ABC$ is tangent to sides $BC, CA$ and $AB$ at $D, E,$ and $F,$ respectively. The incircle of triangle $PBC$ touches the sides $BC, CP,$ and $PB$ at $K, M,$ and $N,$ respectively. Suppose that $Q$ is the point of intersection of lines $EM$ and $FN$. Prove that $A, P, Q$ are collinear if and only if $K$ coincides with $D$.

300. 3 (Huynh Tan Chau) Determine all pairs of integers $(m, n)$ such that
\[ \frac{n}{m} = \frac{(m^2 - n^2)^{n/m} - 1}{(m^2 - n^2)^{n/m} + 1}. \]
300. 4 (Vo Giang Giai, Manh Tu) Prove that if \(a, b, c, d, e \geq 0\) then
\[
\frac{a + b + c + d + e}{5} \geq \sqrt[5]{abcde} + \frac{q}{20},
\]
where \(q = (\sqrt{a} - \sqrt{b})^2 + (\sqrt{b} - \sqrt{c})^2 + (\sqrt{c} - \sqrt{d})^2 + (\sqrt{d} - \sqrt{e})^2\).

306. 1 (Phan Thi Mui) Prove that if \(x, y, z > 0\) and \(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1\) then
\[
(x + y - z - 1)(y + z - x - 1)(z + x - y - 1) \leq 8.
\]

306. 2 (Tran Tuan Anh) Given an integer \(m \geq 4\), find the maximum and minimum values of the expression \(ab^{m-1} + a^{m-1}b\), where \(a, b\) are real numbers such that \(a + b = 1\) and \(0 \leq a, b \leq \frac{m-2}{m}\).

308. 1 (Le Thi Anh Thu) Find all integer solutions of the equation
\[
4(a - x)(x - b) + b - a = y^2,
\]
where \(a, b\) are given integers, \(a > b\).

308. 2 (Phan The Hai) Given a convex quadrilateral \(ABCD\), \(E\) is the point of intersection of \(AB\) and \(CD\), and \(F\) is the intersection of \(AD\) and \(BC\). The diagonals \(AC\) and \(BD\) meet at \(O\). Suppose that \(M, N, P, Q\) are the midpoints of \(AB, BC, CD,\) and \(DA\). Let \(H\) be the intersection of \(OF\) and \(MP\), and \(K\) the intersection of \(OE\) and \(NQ\). Prove that \(HK || EF\).

309. 1 (Vu Hoang Hiep) Given a positive integer \(n\), find the smallest possible \(t = t(n)\) such that for all real numbers \(x_1, x_2, \ldots, x_n\) we have
\[
\sum_{k=1}^{n}(x_1 + x_2 + \cdots + x_k)^2 \leq t(x_1^2 + x_2^2 + \cdots + x_n^2).
\]

309. 2 (Le Xuan Son) Given a triangle \(ABC\), prove that
\[
\sin A \cos B + \sin B \cos C + \sin C \cos A \leq \frac{3\sqrt{3}}{4}.
\]

311. 1 (Nguyen Xuan Hung) The chord \(PQ\) of the circumcircle of a triangle \(ABC\) meets its incircle at \(M\) and \(N\). Prove that \(PQ \geq 2MN\).

311. 2 (Dam Van Nhi) Given a convex quadrilateral \(ABCD\) with perpendicular diagonals \(AC\) and \(BD\), let \(BC\) intersect \(AD\) at \(I\) and let \(AB\) meet \(CD\) at \(J\). Prove that \(BDIJ\) is cyclic if and only if \(AB \cdot CD = AD \cdot BC\).
318. 1 (Dau Thi Hoang Oanh) Prove that if $2n$ is a sum of two distinct perfect square numbers (greater than 1) then $n^2 + 2n$ is the sum of four perfect square numbers (greater than 1).

318. 2 (Nguyen De) Solve the system of equations

\[
x^2(y + z)^2 = (3x^2 + x + 1)y^2z^2,
\]
\[
y^2(z + x)^2 = (4y^2 + y + 1)z^2x^2,
\]
\[
z^2(x + y)^2 = (5z^2 + z + 1)x^2y^2.
\]

318. 3 (Tran Viet Hung) A quadrilateral $ABCD$ is inscribed in a circle such that the circle of diameter $CD$ intersects the line segments $AC, AD, BC, BD$ respectively at $A_1, A_2, B_1, B_2,$ and the circle of diameter $AB$ meets the line segments $CA, CB, DA, DB$ respectively at $C_1, C_2, D_1, D_2.$ Prove that there exists a circle that is tangent to the four lines $A_1A_2,$ $B_1B_2,$ $C_1C_2$ and $D_1D_2.$

319. 1 (Duong Chau Dinh) Prove the inequality

\[
x^2y + y^2z + z^2x \leq x^3 + y^3 + z^3 \leq 1 + \frac{1}{2}(x^4 + y^4 + z^4),
\]
where $x, y, z$ are real non-negative numbers such that $x + y + z = 2.$

319. 2 (To Minh Hoang) Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

\[
2(f(m^2 + n^2))^3 = f^2(m)f(n) + f^2(n)f(m)
\]
for distinct $m$ and $n.$

319. 3 (Tran Viet Anh) Suppose that $AD, BE$ and $CF$ are the altitudes of an acute triangle $ABC.$ Let $M, N,$ and $P$ be the intersection points of $AD$ and $EF, BE$ and $FD, CF$ and $DE$ respectively. Denote the area of triangle $XYZ$ by $F[XYZ].$ Prove that

\[
\frac{1}{F[ABC]} \leq \frac{F[MNP]}{F^2[DEF]} \leq \frac{1}{8 \cos A \cos B \cos C \cdot F[ABC]}.
\]

320. 1 (Nguyen Quang Long) Find the maximum value of the function $f = \sqrt{4x - x^3} + \sqrt{x + x^3}$ for $0 \leq x \leq 2.$

320. 2 (Vu Dinh Hoa) Two circles of centers $O$ and $O'$ intersect at $P$ and $Q$ (see Figure). The common tangent, adjacent to $P$, of the two circles touches $O$ at $A$ and $O'$ at $B$. The tangent of circle $O$ at $P$ intersects $O'$ at $C$, and the tangent of $O'$ at $P$ meets the circle $O$ at $D$. Let $M$ be the reflection of $P$ across the midpoint of $AB$. The line $AP$ intersects $BC$ at $E$ and the line $BP$ meets $AD$ at $F$. Prove that the hexagon $AMBEQF$ is cyclic.
320. 3 (Ho Quang Vinh) Let $R$ and $r$ be the circumradius and inradius of triangle $ABC$: the incircle touches the sides of the triangle at three points which form a triangle of perimeter $p$. Suppose that $q$ is the perimeter of triangle $ABC$. Prove that $r/R \leq p/q \leq \frac{1}{2}$.

321. 1 (Le Thanh Hai) Prove that for all positive numbers $a, b, c, d$

a) \[ \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a + b + c}{\sqrt{abc}}; \]

b) \[ \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{d^2} + \frac{d^2}{a^2} \geq \frac{a + b + c + d}{\sqrt{abcd}}. \]

321. 2 (Pham Hoang Ha) Find necessary and sufficient conditions for which the system of equations

\[ x^2 = (2 + m)y^3 - 3y^2 + my, \]
\[ y^2 = (2 + m)z^3 - 3z^2 + mz, \]
\[ z^2 = (2 + m)x^3 - 3x^2 + mx \]

has a unique solution.

321. 3 (Tran Viet Anh) Let $m, n, p$ be three positive integers such that $n+1$ is divisible by $m$. Find a formula for the set of numbers $(x_1, x_2, \ldots, x_p)$ of $p$ positive primes such that the sum $x_1 + x_2 + \cdots + x_p$ is divisible by $m$, with each number of the set not exceeding $n$.

322. 1 (Nguyen Nhu Hien) Given a triangle $ABC$ with incenter $I$. The lines $AI$ and $DI$ intersect the circumcircle of triangle $ABC$ again at $H$ and $K$, respectively. Draw $IJ$ perpendicular to $BC$ at $J$. Prove that $H, K$ and $J$ are collinear.

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322. 2 (Tran Tuan Anh) Prove the inequality
\[
\frac{1}{2} \left( \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{x_i} \right) \geq n - 1 + \frac{n}{\sum_{i=1}^{n} x_i},
\]
where \( x_i \) (\( i = 1, 2, \ldots, n \)) are positive real numbers such that \( \sum_{i=1}^{n} x_i^2 = n \), with \( n \) as an integer, \( n > 1 \).

323. 1 (Nguyen Duc Thuan) Suppose that \( ABCD \) is a convex quadrilateral. Points \( E, F \) are chosen on the lines \( BC \) and \( AD \), respectively, such that \( AE \parallel CD \) and \( CF \parallel AB \). Prove that \( A, B, C, D \) are concyclic if and only if \( AECF \) has an incircle.

323. 2 (Nguyen The Phiet) Prove that for an acute triangle \( ABC \),
\[
\cos A + \cos B + \cos C + \frac{1}{3}(\cos 3B + \cos 3C) \geq \frac{5}{6},
\]

324. 1 (Tran Nam Dung) Find the greatest possible real number \( c \) such that we can always choose a real number \( x \) which satisfies the inequality \( \sin(mx) + \sin(nx) \geq c \) for each pair of positive integers \( m \) and \( n \).

325. 1 (Nguyen Dang Phat) Given a convex hexagon inscribed in a circle such that the opposite sides are parallel. Prove that the sums of the lengths of each pair of opposite sides are equal if and only if the distances of the opposite sides are the same.

325. 2 (Dinh Van Kham) Given a natural number \( n \) and a prime \( p \), how many sets of \( p \) natural numbers \( \{a_0, a_1, \ldots, a_{p-1}\} \) are there such that
\begin{itemize}
  \item[a)] \( 1 \leq a_i \leq n \) for each \( i = 0, 1, \ldots, p-1 \),
  \item[b)] \( [a_0, a_1, \ldots, a_{p-1}] = p \min\{a_0, a_1, \ldots, a_{p-1}\} \),
\end{itemize}
where \( [a_0, a_1, \ldots, a_{p-1}] \) denotes the least common multiple of the numbers \( a_0, a_1, \ldots, a_{p-1} \)?

327. 1 (Hoang Trong Hao) Let \( ABCD \) be a bicentric quadrilateral (i.e., it has a circumcircle of radius \( R \) and an incircle of radius \( r \)). Prove that \( R \geq r\sqrt{2} \).

327. 2 (Vu Dinh The) Two sequences \( (x_n) \) and \( (y_n) \) are defined by
\[
\begin{align*}
x_{n+1} &= -2x_n^2 - 2x_ny_n + 8y_n^2, \quad x_1 = -1, \\
y_{n+1} &= 2x_n^2 + 3x_ny_n - 2y_n^2, \quad y_1 = 1
\end{align*}
\]
for \( n = 1, 2, 3, \ldots \). Find all primes \( p \) such that \( x_p + y_p \) is not divisible by \( p \).
328.1 (Bui Van Chi) Find all integer solutions \((n, m)\) of the equation 
\[(n + 1)(2n + 1) = 10m^2.\]

328.2 (Nguyen Thi Minh) Determine all positive integers \(n\) such that the polynomial of \(n + 1\) terms 
\[p(x) = x^{4n} + x^{4(n-1)} + \ldots + x^8 + x^4 + 1\]
is divisible by the polynomial of \(n + 1\) terms 
\[q(x) = x^{2n} + x^{2(n-1)} + \ldots + x^4 + x^2 + 1.\]

328.3 (Bui The Hung) Find the smallest possible prime \(p\) such that \([3 + \sqrt{p}]^{2n} + 1\) is divisible by \(2^{n+1}\) for each natural number \(n\), where \([x]\) denotes the integral part of \(x\).

328.4 (Han Ngoc Duc) Find all real numbers \(a\) such that there exists a positive real number \(k\) and functions \(f : \mathbb{R} \to \mathbb{R}\) which satisfy the inequality 
\[f(x) + f(y) \geq f\left(\frac{x+y}{2}\right) + k|x-y|^a,\]
for all real numbers \(x, y\).

328.5 (Vu Hoang Hiep) In space, let \(A_1, A_2, \ldots, A_n\) be \(n\) distinct points. Prove that 
\[a) \sum_{i=1}^{n} \angle A_iA_{i+1}A_{i+2} \geq \pi,\]
\[b) \sum_{i=1}^{n} \angle A_iQA_{i+1} \leq (n-1)\pi,\]
where \(A_{n+1}\) is equal to \(A_1\) and \(Q\) is an arbitrary point distinct from \(A_1, A_2, \ldots, A_n\).

329.1 (Hoang Ngoc Minh) Find the maximum value of the expression 
\[(a - b)^4 + (b - c)^4 + (c - a)^4,\]
for any real numbers \(1 \leq a, b, c \leq 2\).

331.1 (Nguyen Manh Tuan) Let \(x, y, z, w\) be rational numbers such that \(x + y + z + w = 0\). Show that the number 
\[
\sqrt{(xy - zw)(yz - wx)(zx - yw)}
\]
is also rational.
331. 2 (Bui Dinh Than) Given positive reals $a, b, c, x, y, z$ such that
\[ a + b + c = 4 \quad \text{and} \quad ax + by + cz = xyz, \]
show that $x + y + z > 4$.

331. 3 (Pham Nang Khanh) Given a triangle $ABC$ and its angle bisector $AM$, the line perpendicular to $BC$ at $M$ intersects line $AB$ at $N$. Prove that $\angle BAC$ is a right angle if and only if $MN = MC$.

331. 4 (Dao Tam) Diagonals $AC, BD$ of quadrilateral $ABCD$ intersect at $I$ such that $IA = ID$ and $\angle AID = 120^\circ$. From point $M$ on segment $BC$, draw $MN \parallel AC$ and $MQ \parallel BD$, $N$ and $Q$ are on $AB$ and $CD$, respectively. Find the locus of circumcenter of triangle $MNQ$ when $M$ moves on line segment $BC$.

331. 5 (Nguyen Trong Hiep) Let $p, q$ be primes such that $p > q > 2$. Find all integers $k$ such that the equation $(px - qy)^2 = kxyz$ has integer solutions $(x, y, z)$ with $xy \neq 0$.

331. 6 (Han Ngoc Duc) Let a sequence $(u_n), n = 1, 2, 3, \ldots$, be given defined by $u_n = n^{2^n}$ for all $n = 1, 2, \ldots$. Let
\[ x_n = \frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n}. \]
Prove that the sequence $(x_n)$ has a limit as $n$ tends to infinity and that the limit is irrational.

331. 7 (Tran Tuan Anh) Find all positive integers $n \geq 3$ such that the following inequality holds for all real numbers $a_1, a_2, \ldots, a_n$ (assume $a_{n+1} = a_1$)
\[ \sum_{1 \leq i < j \leq n} (a_i - a_j)^2 \leq \left( \sum_{i=1}^{n} |a_i - a_{i+1}| \right)^2. \]

332. 1 (Nguyen Van Ai) Find the remainder in the integer division of the number $a^b + b^a$ by 5, where $a = \overline{22 \ldots 2}$ with 2004 digits 2, and $b = \overline{33 \ldots 3}$ with 2005 digits 3 (written in the decimal system).

332. 2 (Nguyen Khanh Nguyen) Suppose that $ABC$ is an isosceles triangle with $AB = AC$. On the line perpendicular to $AC$ at $C$, let point $D$ such that points $B, D$ are on different sides of $AC$. Let $K$ be the intersection point of the line perpendicular to $AB$ at $B$ and the line passing through the midpoint $M$ of $CD$, perpendicular to $AD$. Compare the lengths of $KB$ and $KD$. 

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332. 3 (Pham Van Hoang) Consider the equation
\[ x^2 - 2kxy^2 + k(y^3 - 1) = 0, \]
where \( k \) is some integer. Prove that the equation has integer solutions \((x, y)\) such that \( x > 0, y > 0 \) if and only if \( k \) is a perfect square.

332. 4 (Do Van Ta) Solve the equation
\[ \sqrt{x - \sqrt{x - \sqrt{x - 5}}} = 5. \]

332. 5 (Pham Xuan Trinh) Show that if \( a \geq 0 \) then
\[ \sqrt{a} + \sqrt{a} + \sqrt{a} \leq a + 2. \]

332. 6 (Bui Van Chi) Let \( ABCD \) be a parallelogram with \( AB < BC \). The bisector of angle \( \angle BAD \) intersects \( BC \) at \( E \); let \( O \) be the intersection point of the perpendicular bisectors of \( BD \) and \( CE \). A line passing through \( C \) parallel to \( BD \) intersects the circle with center \( O \) and radius \( OC \) at \( F \). Determine \( \angle AFC \).

332. 7 (Phan Hoang Ninh) Prove that the polynomial
\[ p(x) = x^4 - 2003x^3 + (2004 + a)x^2 - 2005x + a \]
with \( a \in \mathbb{Z} \) has at most one integer solution. Furthermore, prove that it has no multiple integral root greater than 1.

332. 8 (Phung Van Su) Prove that for any real numbers \( a, b, c \)
\[ (a^2 + 3)(b^2 + 3)(c^2 + 3) \geq \frac{4}{27}(3ab + 3bc + 3ca + abc)^2. \]

332. 9 (Nguyen Van Thanh) Determine all functions \( f(x) \) defined on the interval \((0, +\infty)\) which have a derivative at \( x = 1 \) and that satisfy
\[ f(xy) = \sqrt{x}f(y) + \sqrt{y}f(x) \]
for all positive real numbers \( x, y \).

332. 10 (Hoang Ngoc Canh) Let \( A_1A_2\ldots A_n \) be a \( n \)-gon inscribed in the unit circle; let \( M \) be a point on the minor arc \( A_1A_n \). Prove that
\[
\text{a) } MA_1 + MA_3 + \cdots + MA_{n-2} + MA_n < \frac{n}{\sqrt{2}} \text{ for } n \text{ odd;}
\]
\[
\text{b) } MA_1 + MA_3 + \cdots + MA_{n-3} + MA_{n-1} \leq \frac{n}{\sqrt{2}} \text{ for } n \text{ even.}
\]
When does equality hold?

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332.11 (Dang Thanh Hai) Let $ABC$ be an equilateral triangle with centroid $O$; $\ell$ is a line perpendicular to the plane $(ABC)$ at $O$. For each point $S$ on $\ell$, distinct from $O$, a pyramid $SABC$ is defined. Let $\phi$ be the dihedral angle between a lateral face and the base, let $\gamma$ be the angle between two adjacent lateral faces of the pyramid. Prove that the quantity $F(\phi, \gamma) = \tan^2 \phi [3 \tan^2(\gamma/2) - 1]$ is independent of the position of $S$ on $\ell$.

334.1 (Dang Nhu Tuan) Determine the sum
\[
\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{(n-1)n(n+1)} + \cdots + \frac{1}{23 \cdot 24 \cdot 25}.
\]

334.2 (Nguyen Phuoc) Let $ABC$ be a triangle with angle $A$ not being right, $B \neq 135^\circ$. Let $M$ be the midpoint of $BC$. A right isosceles triangle $ABD$ is outwardly erected on the side $BC$ as base. Let $E$ be the intersection point of the line through $A$ perpendicular to $AB$ and the line through $C$ parallel to $MD$. Let $AB$ intersect $CE$ and $DM$ at $P$ and $Q$, respectively. Prove that $Q$ is the midpoint of $BP$.

334.3 (Nguyen Duy Lien) Find the smallest possible odd natural number $n$ such that $n^2$ can be expressed as the sum of an odd number of consecutive perfect squares.

334.4 (Pham Viet Hai) Find all positive numbers $a, b, c, d$ such that
\[
\frac{a^2}{b+c} + \frac{b^2}{c+d} + \frac{c^2}{d+a} + \frac{d^2}{a+b} = 1 \quad \text{and} \quad a^2 + b^2 + c^2 + d^2 \geq 1.
\]

334.5 (Dao Quoc Dung) The incircle of triangle $ABC$ (incenter $I$) touches the sides $BC, CA, AB$ respectively at $D, E, F$. The line through $A$ perpendicular to $IA$ intersects lines $DE, DF$ at $M, N$, respectively; the line through $B$ perpendicular to $IB$ intersect $EF, ED$ at $P, Q$, respectively; the line through $C$ perpendicular to $IC$ intersect lines $FD, FE$ at $S, T$, respectively. Prove the inequality
\[
MN + PQ + ST \geq AB + BC + CA.
\]

334.6 (Vu Huu Binh) Let $ABC$ be a right isosceles triangle with $A = 90^\circ$. Find the locus of points $M$ such that $MB^2 - MC^2 = 2MA^2$.

334.7 (Tran Tuan Anh) We are given $n$ distinct positive numbers, $n \geq 4$. Prove that it is possible to choose at least two numbers such that their sums and differences do not coincide with any $n - 2$ others of the given numbers.
335.1 (Vu Tien Viet) Prove that for all triangles $ABC$
\[\cos A + \cos B + \cos C \leq 1 + \frac{1}{6} \left( \cos^2 \frac{A - B}{2} + \cos^2 \frac{B - C}{2} + \cos^2 \frac{C - A}{2} \right).\]

335.2 (Phan Duc Tuan) In triangle $ABC$, let $BC = a$, $CA = b$, $AB = c$ and $F$ be its area. Suppose that $M, N,$ and $P$ are points on the sides $BC, CA,$ and $AB$, respectively. Prove that
\[ab \cdot MN^2 + bc \cdot NP^2 + ca \cdot PM^2 \geq 4F^2.\]

335.3 (Tran Van Xuan) In isosceles triangle $ABC$, $\angle ABC = 120^\circ$. Let $D$ be the point of intersection of line $BC$ and the tangent to the circumcircle of triangle $ABC$ at $A$. A line through $D$ and the circumcenter $O$ intersects $AB$ and $AC$ at $E$ and $F$, respectively. Let $M$ and $N$ be the midpoints of $AB$ and $AC$. Show that $AO, MF$ and $NE$ are concurrent.

336.1 (Nguyen Hoa) Solve the following system of equations
\[
\begin{align*}
\frac{a}{x} - \frac{b}{z} &= c - zx, \\
\frac{b}{y} - \frac{c}{x} &= a - xy, \\
\frac{c}{z} - \frac{a}{y} &= b - yz.
\end{align*}
\]

336.2 (Pham Van Thuan) Given two positive real numbers $a, b$ such that $a^2 + b^2 = 1$, prove that
\[\frac{1}{a} + \frac{1}{b} \geq 2\sqrt{2} + \left( \sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}} \right)^2.
\]

336.3 (Nguyen Hong Thanh) Let $P$ be an arbitrary point in the interior of triangle $ABC$. Let $BC = a$, $CA = b$, $AB = c$. Denote by $u, v$ and $w$ the distances of $P$ to the lines $BC, CA, AB$, respectively. Determine $P$ such that the product $uvw$ is a maximum and calculate this maximum in terms of $a, b, c$.

336.4 (Nguyen Lam Tuyen) Given the polynomial $Q(x) = (p-1)x^p - x - 1$ with $p$ being an odd prime number. Prove that there exist infinitely many positive integers $a$ such that $Q(a)$ is divisible by $p^p$.

336.5 (Hoang Minh Dung) Prove that in any triangle $ABC$ the following inequalities hold:
\[
\begin{align*}
a) \quad \cos A + \cos B + \cos C + \cot A + \cot B + \cot C &\geq \frac{3}{2} + \sqrt{3}; \\
b) \quad \sqrt{3} (\cos A + \cos B + \cos C) + \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} &\geq \frac{9\sqrt{3}}{2}.
\end{align*}
\]
337. 1 (Nguyen Thi Loan) Given four real numbers $a, b, c, d$ such that $4a^2 + b^2 = 2$ and $c + d = 4$, determine the maximum value of the expression $f = 2ac + bd + cd$.

337. 2 (Vu Anh Nam) In triangle $ABC$, let $D$ be the intersection point of the internal angle bisectors $BM$ and $CN$, $M$ on $AC$ and $N$ on $AB$. Prove that $\angle BAC = 90^\circ$ if and only if $2BD \cdot CD = BM \cdot CN$.

337. 3 (Tran Tuan Anh) Determine the maximum value of the expression $f = (x - y)(y - z)(z - x)(x + y + z)$, where $x, y, z$ lie in the interval $[0, 1]$.

337. 4 (Han Ngoc Duc) Let $n$, $n \geq 2$, be a natural number, $a, b$ be positive real numbers such that $a < b$. Suppose that $x_1, x_2, \ldots, x_n$ are $n$ real numbers in the interval $[a, b]$. Find the maximum value of the sum $\sum_{1 \leq i < j \leq n} (x_i - x_j)^2$.

337. 5 (Le Hoai Bac) A line through the incenter of a triangle $ABC$ intersects sides $AB$ and $AC$ at $M$ and $N$, respectively. Show that $\frac{MB \cdot NC}{MA \cdot NA} \leq \frac{BC^2}{4AB \cdot AC}$.

338. 1 (Pham Thinh) Show that if $a, b, c, d, p, q$ are positive real numbers with $p \geq q$ then the following inequality holds:

$$\frac{a}{pb + qc} + \frac{b}{pc + qd} + \frac{c}{pd + qa} + \frac{d}{pa + qb} \geq \frac{4}{p + q}.$$ 

Is the inequality still true if $p < q$?

338. 2 (Tran Quang Vinh) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition $f(x^2 + f(y)) = y + xf(x)$ for all real numbers $x, y$.

338. 3 (Tran Viet Anh) Determine the smallest possible positive integer $n$ such that there exists a polynomial $p(x)$ of degree $n$ with integer coefficients satisfying the conditions

a) $p(0) = 1$, $p(1) = 1$;

b) $p(m)$ divided by 2003 leaves remainders 0 or 1 for all integers $m > 0$. 

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338. 4 (Hoang Trong Hao) The Fibonacci sequence \(F_n\), \(n = 1, 2, \ldots\), is defined by \(F_1 = F_2 = 1\), \(F_{n+1} = F_n + F_{n-1}\) for \(n = 2, 3, 4, \ldots\). Show that if \(a \neq F_{n+1}/F_n\) for all \(n = 1, 2, 3, \ldots\) then the sequence \((x_n)\), where
\[x_1 = a, \quad x_{n+1} = \frac{1}{1 + x_n}, \quad n = 1, 2, \ldots\]
is defined and has a finite limit when \(n\) tends to infinity. Determine the limit.

339. 1 (Ngo Van Khuong) Given five positive real numbers \(a, b, c, d, e\) such that \(a^2 + b^2 + c^2 + d^2 + e^2 \leq 1\), prove that
\[\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + cd} + \frac{1}{1 + de} + \frac{1}{1 + ea} \geq \frac{25}{6}.
\]

339. 2 (Le Chu Bien) Suppose that \(ABCD\) is a rectangle. The line perpendicular to \(AC\) at \(C\) intersects lines \(AB, AD\) respectively at \(E, F\). Prove the identity \(BE\sqrt{CF} + DF\sqrt{CE} = AC\sqrt{EF}\).

339. 3 (Tran Hong Son) Let \(I\) be the incenter of triangle \(ABC\) and let \(m_a, m_b, m_c\) be the lengths of the medians from vertices \(A, B\) and \(C\), respectively. Prove that
\[
\frac{IA^2}{m_a^2} + \frac{IB^2}{m_b^2} + \frac{IC^2}{m_c^2} \leq \frac{3}{4}.
\]

339. 4 (Quach Van Giang) Given three positive real numbers \(a, b, c\) such that \(ab + bc + ca = 1\). Prove that the minimum value of the expression \(x^2 + ry^2 + tz^2\) is \(2m\), where \(m\) is the root of the cubic equation \(2x^3 + (r + s + 1)x^2 - rs = 0\) in the interval \((0, \sqrt{rs})\). Find all primes \(r, s\) such that \(2m\) is rational.

339. 5 (Nguyen Truong Phong) The sequence \((x_n)\) is defined by
\[x_n = a_n^{a_n}, \quad \text{where} \quad a_n = \frac{(2n)!}{(n!)^2 \cdot 2^{2n}}, \quad \text{for } n = 1, 2, 3, \ldots.
\]
Prove that the sequence \((x_n)\) has a limit when \(n\) tends to infinity and determine the limit.

339. 6 (Huynh Tan Chau) Let \(a\) be a real number, \(a \in (0, 1)\). Determine all functions \(f : \mathbb{R} \to \mathbb{R}\) that are continuous at \(x = 0\) and satisfy the equation
\[f(x) - 2f(ax) + f(a^2x) = x^2\]
for all real \(x\).
339. 7 (Nguyen Xuan Hung) In the plane, given a circle with center $O$ and radius $r$. Let $P$ be a fixed point inside the circle such that $OP = d > 0$. The chords $AB$ and $CD$ through $P$ make a fixed angle $\alpha$, ($0^\circ < \alpha \leq 90^\circ$). Find the maximum and minimum value of the sum $AB + CD$ when both $AB$ and $CD$ vary, and determine the position of the two chords.

340. 1 (Pham Hoang Ha) Find the maximum value of the expression
\[ \frac{x+y}{1+z} + \frac{y+z}{1+x} + \frac{z+x}{1+y}, \]
where $x, y, z$ are real numbers in the interval $[\frac{1}{2}, 1]$.

340. 2 (Nguyen Quynh) Let $M$ be a point interior to triangle $ABC$, let $AM$ intersect $BC$ at $E$, let $CM$ meet $AB$ at $F$. Suppose that $N$ is the reflection of $B$ across the midpoint of $EF$. Prove that the line $MN$ has a fixed point when $M$ moves in the triangle $ABC$.

340. 3 (Tran Tuan Anh) Let $a, b, c$ be the side lengths of a triangle, and $F$ its area, prove that $F \leq \frac{\sqrt{3}}{4} (abc)^{2/3}$, and determine equality cases.

340. 4 (Han Ngoc Duc) Given non-negative integers $n, k$, $n > 1$ and let $\{a_1, a_2, \ldots, a_n\}$ be the $n$ real numbers, prove that
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{k+i+j} \geq 0. \]

340. 5 (Tran Minh Hien) Does there exist a function $f : \mathbb{R}^* \to \mathbb{R}^*$ such that
\[ f^2(x) \geq f(x+y)(f(x)+y) \]
for all positive real numbers $x, y$?

\[ \ldots \text{to be continued} \]